Pedagogic Aids to Supersymmetry

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These notes are best used with Aitchison (*Supersymmetry in Particle Physics,* Cambridge 2007), but can aid with other texts, as well.

Table of Contents

1 Gauge Hierarchy Fine Tuning Background

NOTE: This Section 1 is an attempt to explain a somewhat obscure part of Aitchison's Chap. 1, from the bottom paragraph of pg. 6 to the middle of pg. 8. However, IMO, it contributes little to an understanding of SUSY, so the reader can skip it (that part in Aitchison and this present Section 1), if she/he chooses, with little impact on the learning process.

1.1 Photons and Loop Corrections

There is no photon mass term in the QED Lagrangian. From Klauber Vol. 1, second line of (11-31), pg. 293

$$\mathcal{L}^{e/m} = \underbrace{-\tfrac{1}{2}\left(\partial^{\nu} A^{\mu}\partial_{\nu} A_{\mu} - \partial^{\nu} A^{\mu}\partial_{\mu} A_{\nu}\right)}_{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}} + \bar{\psi}\left(i\gamma^{\nu}\partial_{\nu} - m\right)\psi + e\bar{\psi}\gamma^{\nu}\psi A_{\nu} \tag{1-1}$$

$$\text{no such term} \quad \rightarrow \quad m_{\gamma}^{2} A^{\mu} A_{\mu} \,. \tag{1-2}$$

free massless photon

Figure 1. Free Photon Feynman Diagram

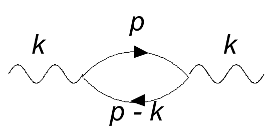
1.1.1 Symmetry of the Photon Terms

In the reference cited before (1-1), after the cited equation, it is shown that the Lagrangian (1-1) is symmetric under $U(1)$ transformations, but would not be symmetric if a term like (1-2) existed.

<u>Bottom line:</u> U(1) symmetry (gauge invariance) means the U(1) gauge field (the photon) is massless.

1.1.2 Higher Order Corrections

This is true at tree level. Higher order corrections might be expected to modify this.



might be equivalent to

where **X** is a mass correction.

Figure 2. One Loop Correction to Free Photon

However, Fig. 2 modifies Fig. 1 because the amplitude for Fig. 2 corrects the amplitude for Fig. 1. But Fig. 1 is not due to terms like (1-2), but to terms like those in the first parentheses after the equal sign in (1-1). There are still no terms like (1-2), even with higher order corrections included.

The divergence from Fig. 2 is absorbed into the running coupling constant $e(p)$, and does not affect the massless nature of the photon.

<u>Bottom line:</u> Symmetry of $\mathcal{L}$ under a U(1) transformation enforces no mass for, and then no mass correction to, the photon.

1.2 Electrons and Loop Corrections

1.2.1 Symmetry and Asymmetry for the Electron Terms

There is a mass term in (1-1), however, for the electron. With this term, the entire $\mathcal{L}$ is symmetric (gauge invariant) under the (local) U(1) transformation. BUT, it is not symmetric under a (global) <u>chiral transformation</u> (in spinor space), i.e., under

$$\psi \;\rightarrow\; \psi' = e^{i\alpha\gamma_5}\psi \quad \text{and} \qquad \bar{\psi} \;\rightarrow\; \bar{\psi}' = \psi'^{\dagger}\gamma^{0} = \psi^{\dagger} e^{-i\alpha\gamma_5^{\dagger}}\gamma^{0} = \psi^{\dagger} e^{-i\alpha\gamma_5}\gamma^{0} \,. \tag{1-3}$$

The mass term in (1-1), transforms under (1-3) as

$$m\bar{\psi}\psi \;\rightarrow\; m\psi^{\dagger} e^{-i\alpha\gamma^5}\gamma^{0} e^{i\alpha\gamma^5}\psi \;\xrightarrow[\gamma^5\gamma^{\mu}=-\gamma^{\mu}\gamma^5]{\text{with}}\; -m\psi^{\dagger}\gamma^{0} e^{-i\alpha\gamma^5} e^{i\alpha\gamma^5}\psi = -m\bar{\psi}\psi \,. \tag{1-4}$$

The mass term is not invariant (it changes sign).

The kinetic electron term in (1-1) is, however, invariant under (1-3).

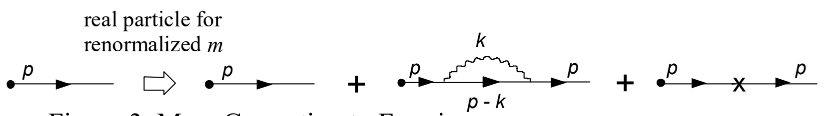
$$\bar{\psi} i\gamma^{\nu}\partial_{\nu}\psi \;\rightarrow\; i\psi^{\dagger}e^{-i\alpha\gamma^5}\gamma^0\gamma^{\nu}e^{i\alpha\gamma^5}\partial_{\nu}\psi \;\xrightarrow[\gamma^5\gamma^{\mu}=-\gamma^{\mu}\gamma^5]{\text{with}}\; i\psi^{\dagger}\gamma^0\gamma^{\nu}e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\partial_{\nu}\psi = \bar{\psi} i\gamma^{\nu}\partial_{\nu}\psi \quad (1\text{-}5)$$

Homework Problem 1-1: Show that the interaction term in (1-1) is invariant under (1-3).

<u>Bottom line:</u> (1-1) is symmetric under a U(1) transformation, but, due to the mass term, is not symmetric under a chiral transformation. However, if the electron were massless, then (1-4) would be zero, and (1-1) would be symmetric under a chiral transformation.

1.2.2 Higher Order Corrections

The kinetic (derivative) term inside the second parentheses after the equal sign in (1-1) is expressed graphically in the left side of Fig. 3 for a free electron. When we incorporate higher order terms mathematically, part of the correction comes out as a loop diagram as in the next to last diagram on the right side of Fig. 3, and another part of the correction manifests as a correction to the mass term, as in the last diagram on the right side of Fig. 3.



X represents δm, the mass correction

Figure 3. Mass Correction to Fermion

Detailed calculation shows the mass correction proportional to mass and the log of Λ, which after regularization is taken to infinity.

$$\delta m \propto \alpha m \ln \Lambda \qquad (1\text{-}6)$$

So, if mass is initially zero, we get no higher order corrections to the mass. For a zero mass term in the Lagrangian, there is zero mass at tree level and at all higher order correction levels. In other words, if the Lagrangian (1-1) is symmetric under a chiral transformation, the electron (or any massless fermion) is massless (at all orders).

<u>Bottom line:</u> Unbroken gauge symmetry keeps gauge vector bosons that are massless at tree level, massless at all higher orders. Unbroken chiral symmetry keeps fermions that are massless at tree level, massless at all levels.

1.3 Impact for SUSY

The symmetries investigated above enforce a "no mass corrections" effect on particles.

We might, therefore, look for some kind of symmetry that groups scalars (like the Higgs) with either massless fermions or massless vector bosons. That could protect the mass of the scalar from ballooning upward due to higher order corrections, which is the gauge hierarchy problem (for the Higgs).

SUSY symmetry is just such a symmetry. It groups the Higgs with fermions (and also the vector bosons with fermions).

1.4 Solution to Homework Problem

Homework Problem 1-1: Show that the interaction term in (1-1) is invariant under (1-3).
Ans.

$$\mathcal{L}_I^{e/m} = e\bar{\psi}\gamma^{\nu}\psi A_{\nu} \qquad (1\text{-}7)$$

From

$$\psi \rightarrow \psi' = e^{i\alpha\gamma_5}\psi \quad \text{and} \quad \bar{\psi} \rightarrow \bar{\psi}' = \psi'^{\dagger}\gamma^0 = \psi^{\dagger}e^{-i\alpha\gamma_5^{\dagger}}\gamma^0 = \psi^{\dagger}e^{-i\alpha\gamma_5}\gamma^0, \qquad \text{repeat of (1-3)}$$

we have

$$\begin{aligned} e\bar{\psi}\gamma^{\nu}\psi A_{\nu} \;\rightarrow\; e\bar{\psi}'\gamma^{\nu}\psi' A'_{\nu} &= e\psi^{\dagger}e^{-i\alpha\gamma^5}\gamma^0\gamma^{\nu}e^{i\alpha\gamma^5}\psi A_{\nu} = -e\psi^{\dagger}e^{-i\alpha\gamma^5}\gamma^0 e^{i\alpha\gamma^5}\gamma^{\nu}\psi A_{\nu} \\ &= e\psi^{\dagger}e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\gamma^0\gamma^{\nu}\psi A_{\nu} = e\psi^{\dagger}\gamma^0\gamma^{\nu}\psi A_{\nu} = e\bar{\psi}\gamma^{\nu}\psi A_{\nu}. \end{aligned} \qquad (1\text{-}8)$$

2 Gauge Hierarchy and SUSY Cancellations

2.1 Higgs Field in the Standard Model

2.1.1 Review of Higgs Symmetry Breaking

Before symmetry breaking, the free Higgs terms in the Lagrangian[1] are

$$\mathcal{L}_{Higgs} = \partial_\mu \phi \partial^\mu \phi + \left|\mu^2\right| \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 \qquad \text{(high energy)} \tag{2-1}$$

For a complex scalar field, the coefficient of $-\phi^\dagger \phi$ represents the mass squared. For the high energy Higgs ϕ, the mass squared at high energy (before breaking) would therefore be

$$m_\phi^2 = -\left|\mu^2\right| \qquad \text{(high energy Higgs "mass" squared; } m_\phi \text{ imaginary).} \tag{2-2}$$

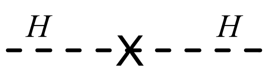
Generally, however, we think of (2-1) as negative of the potential (which shows up in the Lagrangian) for the field ϕ, and all particles as massless. All SM particle masses, after symmetry breaking, depend on μ and λ.

After symmetry breaking, we deal with a low energy Higgs, designated herein by σ, which turns out to be a real (not complex) scalar field. (See Klauber (2021) Chap. 7, summarized in Wholeness Chart 7-8, pg. 240). The mass squared of that field is

$$M_H^2 = 2\left|\mu^2\right| \qquad \text{(low energy Higgs mass squared; } M_H \text{ is real and positive).} \tag{2-3}$$

The low energy Higgs mass differs from the high energy "mass" of (2-2) by the factor $i\sqrt{2}$.

In a Feynman diagram, mass is represented by X. For dashed lines representing the Higgs, this looks like Fig. 1, which has external Higgses in and out. The term in the Lagrangian for this (at low energy) is shown in (2-4).



$$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma = -\left|\mu^2\right|\sigma\sigma \qquad \mathcal{H}_{\sigma\, mass} = \frac{M_H^2}{2}\sigma\sigma = \left|\mu^2\right|\sigma\sigma \tag{2-4}$$

Figure 1. Feynman Diagram Representing the Higgs Mass

Note that the mass squared term in the Lagrangian changes sign as energy falls below the symmetry breaking scale. We can determine $|\mu^2|$ (theoretically, or in principle, experimentally) at high energy or at low. In either case, it will be the coefficient of the Higgs field bi-linear term (either $\phi^\dagger \phi$ or $\sigma\sigma$), just with different signs in each.

The expectation value (what we measure, symbolized by an overbar) for $|\mu^2|$ is, where we recognize that σ creates and destroys states with a single low energy Higgs particle,

$$\overline{\left|\mu^2\right|} = \langle\sigma|\left(\left|\mu^2\right|\sigma\sigma\right)|\sigma\rangle = \left|\mu^2\right|\langle\sigma|\sigma|0\rangle = \left|\mu^2\right|\langle\sigma|\sigma\rangle = \left|\mu^2\right|. \tag{2-5}$$

2.1.2 Our Goal: Determine Radiative Corrections to the Higgs Mass

We are looking to find modifications to the contemporary Higgs mass M_H, or, due to (2-3), $|\mu|$, caused by radiative corrections. Radiative corrections typically entail integrations over particle loops, where such integrations are carried out in principle from $-\infty$ to $+\infty$ energy levels inside the loop. However, practically, one considers the energy levels to be restricted to values below the Planck scale. In either case, the integration limits are taken as $-\Lambda$ to $+\Lambda$, where Λ can be taken as infinite, or as the Planck energy, or as some other yet to be discovered level at which our physics changes and the $\mathcal{L}$ we are using is no longer valid.

(2-1) plus interaction terms in the high energy (false vacuum) Lagrangian (Klauber (2021) (6-48) pg. 176) result, after symmetry breaking, into terms in the low energy (true vacuum) Lagrangian, (Klauber (2021) pg.251 with Feynman rules on pgs. 290-293). We will be using these Feynman rules in what follows.

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[1] μ^2 is taken as positive (real μ) in Aitshison (2007) with $+\mu^2\phi^\dagger\phi$ appearing in the Lagrangian. In Klauber Vol. 2 μ^2 is taken as negative with $-\mu^2\phi^\dagger\phi$ appearing in $\mathcal{L}$. We hopefully avoid confusion by using the term $+|\mu^2|\phi^\dagger\phi$ in (2-1). Also, Aitchison uses $\lambda/4$, where Klauber uses λ. We opt for the later choice, as we feel it simpler. Appendix A summarizes of the differences in the two approaches.

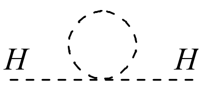
As the reader familiar with QFT should be aware, loop integrations typically result in divergences, to different degrees (quadratic, linear, logarithmic) in the integration limit Λ, as that limit tends to infinity. We will examine corrections to $|\mu|$, and thus to M_H, from each of these, starting in the next section with the quadratic ones, the most severe.

2.2 Higgs Mass Divergences

2.2.1 Quadratic Divergences

4 Higgs Vertex Loop (Quadratic Contribution)

The $-\lambda\left(\phi^{\dagger}\phi\right)^{2}$ term in (2-1), part of the high energy Lagrangian, manifests in the low energy Lagrangian as $-\frac{\lambda}{4}(\sigma\sigma)^{4}$, and represents a four-particle vertex *HHHH* as shown in Feynman rule #15-17 of the above cited reference. Two of the σ fields can result in a propagator loop as illustrated in Fig. 2. Note that we have an external Higgs in and an external Higgs out, as we did in Fig. 1 and (2-5). So, we need to add the amplitude associated with Fig. 2 to that of (2-5) , to find the physical value we would measure for $|\mu^2|$. The vertex factor of the afore cited Feynman rule and the scalar propagator give us (2-6), where λ is known from experiment to be approximately ½,



$$-\left(-i6\lambda\int\frac{i}{k^{2}-m^{2}+i\varepsilon}d^{4}k\right)\sigma\sigma\approx-\left(6\lambda\int_{0}^{\Lambda}\frac{1}{k^{2}}2\pi^{3}k^{3}dk\right)\sigma\sigma=-6\pi^{3}\lambda\Lambda^{2}\sigma\sigma \qquad (2\text{-}6)$$

Figure 2. SM Higgs λ Loop

(2-6) contributes a quadratic radiative correction to (2-5). Note that although at low energy, $-\left|\mu^{2}\right|\sigma\sigma$ in (2-5) and $-\frac{\lambda}{4}(\sigma\sigma)^{4}$ (which gives rise to (2-6)), have the same signs, at high energy (see (2-1)), the terms have opposite signs. We are concerned with the high energy corrections to $|\mu^2|$, so we will consider the difference between (2-5) and (2-6), rather than the sum. Thus, (2-5) is corrected to become

$$\overline{\left|\mu^{2}\right|}=\left\langle\sigma\right|\left(\left|\mu^{2}\right|-6\pi^{3}\lambda\Lambda^{2}\right)\sigma\sigma\left|\sigma\right\rangle=\left|\mu^{2}\right|-6\pi^{3}\lambda\Lambda^{2}. \qquad (2\text{-}7)$$

We can consider the RHS of (2-7) as representing a "physical" μ value, which includes the original μ term plus the radiative correction term, and represents what we would measure at high energy from including both in the Lagrangian. That is,

$$\left|\mu_{phys}^{2}\right|=\left|\mu^{2}\right|-6\pi^{3}\lambda\Lambda^{2}. \qquad (2\text{-}8)$$

Essentially, (2-8) replaces $|\mu^2|$ in (2-1), and hence, our Higgs mass diverges quadratically with Λ. This effect cascades into all SM particle masses, as they depend on μ (now μ_{phys}).

$$\mathcal{L}_{Higgs}=\partial_{\mu}\phi\partial^{\mu}\phi+\left|\mu_{phys}^{2}\right|\phi^{\dagger}\phi-\lambda\left(\phi^{\dagger}\phi\right)^{2}=\partial_{\mu}\phi\partial^{\mu}\phi+\left|\mu^{2}\right|-6\pi^{3}\lambda\Lambda^{2}-\lambda\left(\phi^{\dagger}\phi\right)^{2} \qquad (2\text{-}9)$$

This implies, assuming Λ is merely large and not infinite, that not only the Higgs mass, but all SM particle masses should be on the order of Λ, i.e., far higher than the SM mass scale. This is true unless the two terms in (2-8) are of the same order. But since μ_{phys} is known to be on the order of the weak scale (10^2 GeV) and Λ is considered to be on the order of the Plank scale (10^{18} GeV), an extraordinary fine tuning would be required. That is, μ would have to comprise 17 or so digits, yet only differ from Λ in the last digit. This is the hierarchy problem, or at least, the first part of the hierarchy problem. (We need to include further radiative corrections to μ to define it completely.)

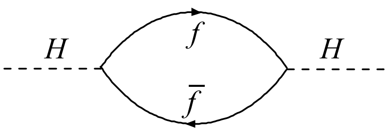
Note two things. First, (2-8) implies, for $\left|\mu^{2}\right|<6\pi^{3}\lambda\Lambda^{2}$, that the actual coefficient of $\phi^{\dagger}\phi$ in (2-1) is negative, turning the Mexican hat diagram for the Higgs potential energy density into a bowl diagram. That means no symmetry breaking, as the Higgs field would then be in a stable configuration. That doesn't work.

Second, if there were new physics at a lower scale than Planck, it would mean our naïve Feynman diagrams and amplitudes should not be integrated to a Plank scale energy, but something lower. If that scale were low enough, then (2-8) would not be so problematic. The $|\mu^2|$ and $\lambda\Lambda^2$ terms could be in the same ballpark, and (2-8) would not give us a hierarchy problem.

Fermion Loops (Quadratic Contribution)

There are other contributions to μ_{phys}, however, that also need to be considered. Any fermion f can form loops with the Higgs like that shown in Fig. 3. See Klauber Vol. 2, pgs 292-293, Feynman rules #15-13 to #15-16, for the particular fermion f. The vertex factor, ignoring some subtleties for neutrinos (whose mass oscillates), is shown there to be, where g_f is the Yukawa coupling and m_f is the fermion mass,

$$\text{vertex factor for Fig. 3} \quad \frac{-i}{\sqrt{2}} g_f = \frac{-i}{v} m_f \qquad v = \frac{\left|\mu_{phys}\right|}{\sqrt{\lambda}} \text{ from symmetry breaking theory.} \quad \text{(2-10)}$$



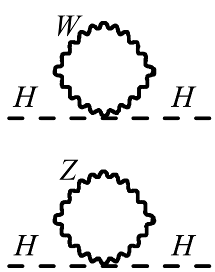
$$\left(-\left(\frac{-i}{\sqrt{2}} g_f \right)^2 \int Tr \left(\frac{i}{\left(\left(\not{k}_H - \not{p} \right) - m \right) + i\varepsilon} \right) \left(\frac{i}{\left(\not{p} - m \right) + i\varepsilon} \right) d^4 p \right) \sigma\sigma$$
$$\approx \frac{g_f^2}{2} \int_0^\Lambda Tr \left(\frac{1}{p^2} \right) 2\pi^3 p^3 dp \sigma\sigma = \pi^3 g_f^2 \int_0^\Lambda Tr \left(I_{4X4} \right) p dp \phi_H \phi_H = 2\pi^3 g_f^2 \Lambda^2 \sigma\sigma \quad \text{(2-11)}$$

Figure 3. SM Fermion Loop

Each possible fermion must be considered, so (2-11) must be summed over f. This result is another quadratic divergence (a series of them, actually) that must be added to (2-7) and (2-8), further modifying our value of μ_{phys}. Before doing so, however, we need to consider yet other quadratic contributions, shown in the following sub-sections.

Gauge Boson 4-Vertex Loops (Quadratic Contributions)

Gauge boson loops also make quadratic contributions. See Klauber Vol. 2, pgs. 292-293, Feynman rules #15-19 and #15-21 for the vertex factors associated with Fig. 4.



$$\text{For } W \text{ loop:} \quad \left(\frac{i}{2} g^2 g^{\mu\nu} \int_0^\Lambda \frac{-ig_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 - m^2 + i\varepsilon} d^4 k \right) \sigma\sigma$$
$$\xrightarrow[\text{propagator acts like massless}]{\text{high energy range}} \approx \left(\frac{g^2}{2} \int_0^\Lambda \frac{4}{k^2} 2\pi^3 k^3 dk \right) \sigma\sigma = 2g^2 \pi^3 \Lambda^2 \sigma\sigma \quad \text{(2-12)}$$

$$Z \text{ loop:} \quad \left(\frac{i}{2 cos^2 \theta_W} g^2 g^{\mu\nu} \int_0^\Lambda \frac{-ig_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 - m^2 + i\varepsilon} d^4 k \right) \sigma\sigma \approx \frac{2g^2 \pi^3}{cos^2 \theta_W} \Lambda^2 \sigma\sigma \quad \text{(2-13)}$$

Figure 4. SM 4-Boson Vertex Loop

So, we'll need to add contributions from (2-12) and (2-13) to (2-8), along with (2-11), *en route* to determining μ_{phys}, the radiatively corrected μ.

Tadpole Divergences

In a non-renormalized theory, quadratic divergences also arise from amplitudes for what are known as tadpole diagrams. However, in a renormalized theory, which the SM is, tadpole diagram interactions make no contributions and can be ignored. So, we ignore them here.

Nature of Divergences

Note that all of the divergences shown in Figs. 2 to 4 are quadratic, i.e., have energy scale Λ^2. Together these give us a quadratically corrected $|\mu^2|$

$$\text{from quadratic terms} \quad \left| \mu_{phys}^2 \right| = \left| \mu^2 \right| \overbrace{- 6\pi^3 \lambda \Lambda^2}^{HHHH} + \overbrace{\sum_f 2\pi^3 g_f^2 \Lambda^2}^{HHf + HH\bar{f}} + \overbrace{2g^2 \pi^3 \Lambda^2}^{HHWW} + \overbrace{\frac{2g^2 \pi^3}{cos^2 \theta_W} \Lambda^2}^{HHZZ}$$
$$= \left| \mu^2 \right| - \left(6\pi^3 \lambda - \sum_f 2\pi^3 g_f^2 - 2g^2 \pi^3 \left(1 + \frac{1}{cos^2 \theta_W} \right) \right) \Lambda^2 . \quad \text{(2-14)}$$

We know μ_{phys} from experiment to be about 88 GeV ($\mu_{phys} = M_H / \sqrt{2} \approx 125/\sqrt{2}\text{GeV}$), with order on that of the weak scale ~10^2. And in the SM, we have no intermediate new physics (no symmetry breaking) between the weak and Planck scales, with the latter ~10^{18}. So, the order of magnitudes in (2-14) can be expressed as

$$\underbrace{\left|\mu_{phys}^2\right|}_{\sim 10^4} = \left|\mu^2\right| - \underbrace{\left(\underbrace{6\pi^3\lambda}_{\sim 10^2} - \underbrace{\sum_f 2\pi^3 g_f^2}_{\sim 10^1} - \underbrace{2g^2\pi^3\left(1+\frac{1}{cos^2\theta_W}\right)}_{\sim 10^1} \right)\underbrace{\Lambda^2}_{\sim 10^{38}}}_{\text{Call this } X^2\text{, which has order } \sim 10^{40} \text{ and is positive}} . \qquad (2\text{-}15)$$

In order to get the μ_{phys} we see in nature, the constant $|\mu^2|$ has to be greater than X^2, but equal it to up the 36th digit. Only then does subtracting the two give a quantity of order 10^4. Put another way, nature had to pick a constant $|\mu|$ that matches the radiative corrections to 18 digits, in order to give a μ_{phys} of order 10^2. This, to say the least, is unnatural.

In other words, we would expect μ_{phys}, and thus the contemporary Higgs mass M_H, to be on the order of the Planck scale, due to the huge radiative corrections. But it is not. This is the biggest part of the gauge hierarchy problem in all its (infamous) glory.

2.2.2 Linear Divergences

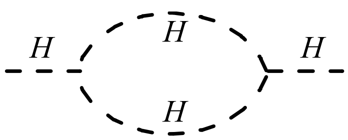
There are no linear divergences for radiative corrections to the Higgs mass – no terms with factors of just Λ.

2.2.3 Logarithmic Divergences

There are logarithmic divergences, as we show in the following.

3 Higgs Vertex Loop (Log Contribution)

Higgs 3 vertex terms lead to additional corrections dependent on the logarithm of energy level. See above cited reference, Feynman rule #15-18, where the vertex factor is $-i6\lambda v$.



$$\left((-i6\lambda v)^2 \int \left(\frac{i}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right)\left(\frac{i}{k^2 - m^2 + i\varepsilon} \right) d^4k \right)\sigma\sigma$$

$$\approx \left(36\lambda^2 v^2 \int_0^\Lambda \frac{1}{k^4} 2\pi^3 k^3 dk \right)\sigma\sigma = 72\pi^3\lambda^2 \frac{\left|\mu_{phys}\right|^2}{\lambda} ln\Lambda\sigma\sigma \qquad (2\text{-}16)$$

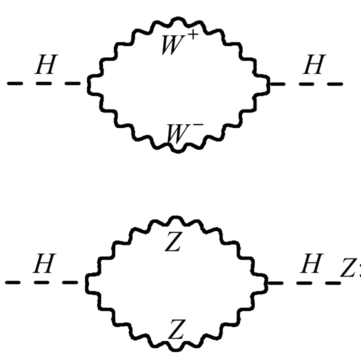
$$= 72\pi^3\lambda \frac{M_H^2}{4} ln\Lambda\sigma\sigma = 18\pi^3\lambda M_H^2 \, ln\Lambda\sigma\sigma$$

Figure 5. SM 3 Higgs Vertex Loop

Note this correction varies with the log of Λ, rather than its square, unlike (2-6), (2-11), (2-12) and (2-13).

Gauge Bosons with Higgs 3-Vertex Loops (Log Contributions)

The appropriate vertex relations represented in Fig. 6 are shown in the above cited reference as Feynman rules #15-20 and #15-22. The resulting amplitudes are logarithmically divergent.



$$W \text{ loop: } \left(\left(\frac{i}{2}g^2 v\right)^2 \int \left(\frac{-g_{\mu\nu} + k_\mu k_\nu / m^2}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right)\left(\frac{-g^{\mu\nu} + k^\mu k^\nu / m^2}{k^2 - m^2 + i\varepsilon} \right) d^4k \right)\sigma\sigma$$

$$\approx -\left(\frac{g^4 v^2}{4} \int_0^\Lambda \frac{4}{k^4} 2\pi^3 k^3 dk \right)\sigma\sigma = -\, 2g^4 \frac{\left|\mu_{phys}^2\right|}{\lambda}\pi^3 \, ln\Lambda\sigma\sigma = -\pi^3 g^4 \frac{M_H^2}{\lambda} ln\Lambda\sigma\sigma \qquad (2\text{-}17)$$

$$Z\text{: } \left(\left(\frac{i}{2cos^2\theta_W}g^2 v\right)^2 \int \left(\frac{-g_{\mu\nu} + k_\mu k_\nu / m^2}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right)\left(\frac{-g^{\mu\nu} + k^\mu k^\nu / m^2}{k^2 - m^2 + i\varepsilon} \right) d^4k \right)\sigma\sigma$$

$$\approx -\left(\frac{g^4 v^2}{4cos^4\theta_W} \int_0^\Lambda \frac{4}{k^4} 2\pi^3 k^3 dk \right)\sigma\sigma = -\pi^3 \frac{g^4}{cos^4\theta_W} \frac{M_H^2}{\lambda} ln\Lambda\sigma\sigma \qquad (2\text{-}18)$$

Figure 6. SM 2 Bosons with Higgs Vertex Loop

2.2.4 Impact of Logarithmic Divergences

Log corrections to μ are nowhere near as impactful as quadratic divergences. Indeed, 10^{19} is quite a different animal from $\ln 10^{19} \approx 44$. Nevertheless, the log terms contribute, when one considers the other factors the log factor is multiplied by, on the order of 10^4 to $|\mu_{phys}|$. As we learned in QED, there are ways to tame log corrections (by renormalization in QED, for example). However, there are circumstances, such as GUTs, where they can become problematic.

In general, the primary concern is with quadratic divergences, but we need also need to keep an eye on the logarithmic ones, whose impact is, relatively speaking, far less, but nevertheless can be difficult to reconcile.

2.2.5 Physical μ with Quadratic plus Logarithmic Corrections

For all corrections, quadratic plus logarithmic, up to second order, we need to add (2-16), (2-17), and (2-18) to (2-14)

$$\left|\mu_{phys}^2\right| = \left|\mu^2\right| - \left(6\pi^3\lambda - \sum_f 2\pi^3 g_f^2 - 2g^2\pi^3\left(1+\frac{1}{\cos^2\theta_W}\right)\right)\Lambda^2 + M_H^2\pi^3\left(18\lambda - \frac{g^4}{\lambda}\left(1+\frac{1}{\cos^4\theta_W}\right)\right)\ln\Lambda. \quad (2\text{-}19)$$

Note that the first log term is positive, and as one sees after plugging in numbers for the various constants, is an order of magnitude greater than the final two log terms combined. So, the log terms add to $|\mu|^2$, whereas the quadratic ones subtract from it. All in all, a mess. And all together, the gauge hierarchy problem.

2.2.6 Other Considerations

2.2.7 Corrections to Higher Order

We have, of course, only dealt with second order corrections up to here. For a complete analysis, one would have to include higher order corrections, as well. Any solution to the hierarchy problem would need to handle corrections at all orders.

2.2.8 Low vs High Energy Analysis

We have worked with the low energy (after symmetry breaking) Feynman rules, with associated low energy propagators. These propagators have mass terms in them, but above the electroweak symmetry breaking scale, particles have no mass. In the loop integration analyses herein, however, we ignored the mass terms because they became insignificant over higher energies, which comprised, for all practical purposes, virtually the entire integration range. Additionally, contributions from portions of the integral over high energy dwarf, by an enormous amount, the contributions from lower energy.

Further, we used propagator expressions for low energy weak boson gauge fields W and Z, rather than for the high energy $SU(2)$ and $U(1)$ vector gauge fields W^1, W^2, W^3, and B. However, since the former fields are but linear combinations of the latter, the result either way is the same, i.e., (2-19).

2.3 Naïve Power Counting vs Proper Regularization

We have used the cutoff regularization method (see Klauber (2013) Chap. 15, summarized in Wholeness Chart 15-2, pg. 397) to evaluate the divergent integrals. But as noted in the cited reference, the cut-off method doesn't work, primarily because it is not Lorentz nor gauge invariant. (See the cited reference for details.) In fact, for interactions like that of Fig. 3, the true divergence turns out to be logarithmic, rather than quadratic, which the naïve power counting of the cutoff method gives us.

This brings up the non-trivial concern that the quadratic divergences cited above, and the concomitant position that the Higgs mass should be of Planck scale energy, may not, in the final and correct analysis, be valid. This position is to be found throughout the literature, but one familiar with photon self-energy in QED will have serious difficulty buying into it.

Nevertheless, for the time being, we proceed in the following with the traditional rendition of gauge hierarchy.

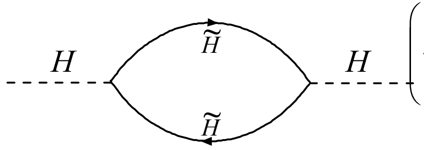
2.4 Supersymmetry and Gauge Hierarchy

2.4.1 Supersymmetry Contributions to Higgs Mass Corrections

As the reader should be aware, supersymmetry (SUSY) adds a SUSY boson (sfermion) for every SM fermion and a fermion (bosino) for every SM boson. Couplings (constants and interactions) for the SUSY spartners mirror those of the SM partners. Masses of spartners equal those of their SM partners above some presumed SUSY breaking scale.

2.4.2 Higgsino Loop

Consider $\tilde{H}$ as a Higgsino (fermionic), where $M_{\tilde{H}} = M_H$ (see Appendix B for final relation)

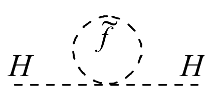


$$\left(-\left(\frac{-i}{\sqrt{2}}g_{H\tilde{H}\tilde{H}}\right)^2\int_0^\Lambda Tr\left(\frac{1}{((\not{k}-\not{p})-m)+i\varepsilon}\right)\left(\frac{1}{(\not{p}-m)+i\varepsilon}\right)d^4p\right)\phi^\dagger\phi \approx \left(\frac{g^2_{H\tilde{H}\tilde{H}}}{2}\int_0^\Lambda Tr\left(\frac{1}{p^2}\right)2\pi^3p^3\,dp\right)\phi^\dagger\phi = 2\pi^3 g^2_{H\tilde{H}\tilde{H}}\Lambda^2\phi^\dagger\phi \tag{2-20}$$

Figure 7. SUSY Higgsino Loop

2.4.3 Sfermion Loops

There are other contributions to μ_{phys}, however, such as the fermion loops of Fig. 3, which manifest as the sum over f term in (2-24). These might be cancelled by other SUSY amplitudes/diagrams, such as that shown in Fig. 8, where $\tilde{f}$ is bosonic, a SUSY partner of a standard model fermion. We get a similar result as in (2-6) but with a different constant $\lambda_{HH\tilde{f}\tilde{f}}$ in place of λ.

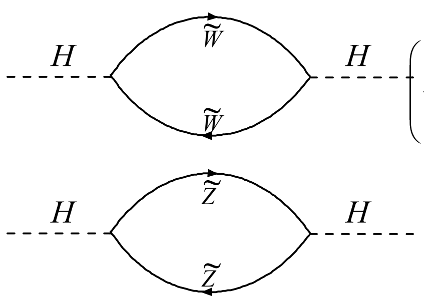


$$-\left(-i6\lambda_{HH\tilde{f}\tilde{f}}\int\frac{i}{k^2-m^2+i\varepsilon}d^4k\right)\sigma\sigma \approx -\left(6\lambda_{HH\tilde{f}\tilde{f}}\int_0^\Lambda\frac{1}{k^2}2\pi^3k^3\,dk\right)\sigma\sigma = -6\pi^3\lambda_{HH\tilde{f}\tilde{f}}\Lambda^2\sigma\sigma \tag{2-21}$$

Figure 8. SUSY Sfermion Loop

2.4.4 Gaugino Loops

In SUSY one might have loops of winos and zinos, as shown in Fig. 9, with concomitant contributions to the Higgs mass as in (2-22) and (2-23).



$$\left(-\left(\frac{-i}{\sqrt{2}}g_{H\tilde{W}\tilde{W}}\right)^2\int_0^\Lambda Tr\left(\frac{1}{((\not{k}-\not{p})-m)+i\varepsilon}\right)\left(\frac{1}{(\not{p}-m)+i\varepsilon}\right)d^4p\right)\phi^\dagger\phi \approx 2\pi^3 g^2_{H\tilde{W}\tilde{W}}\Lambda^2\phi^\dagger\phi \tag{2-22}$$

$$\approx 2\pi^3 g^2_{H\tilde{Z}\tilde{Z}}\Lambda^2\phi^\dagger\phi \tag{2-23}$$

Figure 9. Weak Gaugino Loops

2.4.5 Adding SUSY Terms to Mass Correction

The $|\mu^2|$ factor of the $\phi^\dagger\phi$ term in (2-1) is thus, in SUSY, modified by adding (2-20), (2-21), (2-22), and (2-23) to (2-19).

$$
\begin{aligned}
\left|\mu^2_{phys}\right| = \left|\mu^2\right| - \Bigg(& \overbrace{3\lambda}^{HHHH} \overbrace{- g^2_{H\tilde{H}\tilde{H}}}^{H\tilde{H}\tilde{H}+H\tilde{H}\tilde{H}} \overbrace{-\sum_f g_f^2}^{Hff+Hff} + \overbrace{\sum_{\tilde{f}} 3\lambda_{HH\tilde{f}\tilde{f}}}^{HH\tilde{f}\tilde{f}} - g^2 \Bigg(\overbrace{1}^{HHWW} + \overbrace{\frac{1}{\cos^2\theta_W}}^{HHZZ} \Bigg) \overbrace{+g^2_{H\tilde{W}\tilde{W}}}^{H\tilde{W}\tilde{W}+H\tilde{W}\tilde{W}} \overbrace{+g^2_{H\tilde{Z}\tilde{Z}}}^{H\tilde{Z}\tilde{Z}+H\tilde{Z}\tilde{Z}} \Bigg) 2\pi^3\Lambda^2 \leftarrow \text{quadratic terms} \\
& + \Bigg(\underbrace{18\lambda}_{HHH+HHH} - \frac{g^4}{\lambda} \Bigg(\underbrace{1}_{HWW+HWW} + \underbrace{\frac{1}{\cos^4\theta_W}}_{HZZ+HZZ} \Bigg) \Bigg) M_H^2 \pi^3 \ln\Lambda . \leftarrow \text{log terms}
\end{aligned}
$$

. (2-24)

If $3\lambda = g^2_{H\tilde{H}\tilde{H}}$, we would get cancellation of the first two divergent terms in the top row of (2-24). That is, Fig. 7 would cancel Fig. 2. If $3\lambda_{HH\tilde{f}\tilde{f}} = g_f^2$, the next two divergent terms would cancel. That is, Fig. 8 would cancel Fig. 3. If $g = g_{H\tilde{W}\tilde{W}}$ and $g = \cos\theta_W \, g_{H\tilde{Z}\tilde{Z}}$, then the last terms in (2-24) would also cancel. In SUSY, one finds, after considerable analysis, that this can indeed be true. The signs in (2-20) to (2-23) are critical for this cancellation.

We still have log terms to worry about, but one can see that cancellation of terms can occur in SUSY that could eliminate, or at least ameliorate, the gauge hierarchy problem.

2.5 Problems?

In the standard model, there are no vertices like that of Fig. 8 for SM particles, i.e., no *HHff* vertices. So, why should one expect vertices in SUSY like $HH\tilde{f}\tilde{f}$? See Klauber, Vol. 2, pgs 292-293, 251, and 176, for a summary of Higgs vertices in the SM.

And as mentioned in Sect. 2.3, divergences, such as the fermion loop in boson propagation are naively quadratic, and that is assumed herein, but when fully evaluated, are only logarithmically divergent.

2.6 Appendix A. Comparison of Symmetry Breaking in Aitchison and Klauber

See Wholeness Chart 2-1 on the next page.

Wholeness Chart 2-1. Symmetry Breaking in Aitchison vs Klauber

ϕ is high energy Higgs doublet (complex). σ is low energy singlet (real)

Entity	Typical Scalar	Higgs Scalar in Klauber	Higgs Scalar in Aitchison	Difference/Comment
$\phi^\dagger\phi$ Part of Potential	$\mathcal{V}_{mass} = m^2\phi^\dagger\phi$	$\mathcal{V}_{mass+\lambda} = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2 = -\left\|\mu^2\right\|\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$	$\mathcal{V}_{mass+\lambda} = -\mu^2\phi^\dagger\phi + \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$	$\mu_{Kl}^2 < 0 \quad \mu_{Atich}^2 > 0$ $\lambda_{Kl} = \frac{\lambda_{Aitch}}{4}$
$\phi^\dagger\phi$ Part of Lagrangian	$\mathcal{L}_{mass} = -m^2\phi^\dagger\phi$	$\mathcal{L}_{mass+\lambda} = -\mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 = \left\|\mu^2\right\|\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2$	$\mathcal{L}_{mass+\lambda} = \mu^2\phi^\dagger\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$	ϕ is complex $m = \mu$ (nat. units)
Mass, high energy	$m^2 > 0$ m real	μ = high energy mass $\mu^2 < 0 \quad \mu$ imaginary	$i\mu$ = high energy mass $\mu^2 > 0 \quad \mu$ real	
Mass, low energy	$M^2 > 0$ M real > 0	M_H = low energy mass $M_H = \sqrt{-2\mu^2} = \sqrt{2}\left\|\mu\right\|$	M_H = low energy mass $M_H = \sqrt{2\mu^2} = \sqrt{2}\mu = \sqrt{2}\left\|\mu\right\|$	unitary gauge
Mass term, low energy	$\mathcal{L}_{mass} = -\frac{M^2}{2}\phi\phi$	$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma = -\left\|\mu^2\right\|\sigma\sigma$	$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma = -\left\|\mu^2\right\|\sigma\sigma$	σ is real
ϕ_{min}	N/A	$\phi_{min} = v = \sqrt{\frac{-\mu^2}{\lambda_{Kl}}} = \frac{\left\|\mu\right\|}{\sqrt{\lambda_{Kl}}}$	$\phi_{min} = \frac{v}{\sqrt{2}} = \sqrt{\frac{2\mu^2}{\lambda_{Atich}}} = \frac{\sqrt{2}\left\|\mu\right\|}{\sqrt{\lambda_{Atich}}}$	
v	N/A	$v = \frac{\left\|\mu\right\|}{\sqrt{\lambda_{Kl}}}$	$v = \frac{2\left\|\mu\right\|}{\sqrt{\lambda_{Atich}}} = \frac{\left\|\mu\right\|}{\sqrt{\lambda_{Atich}/4}} = \frac{\left\|\mu\right\|}{\sqrt{\lambda_{Kl}}}$	v is same for both; ϕ_{min} defined differently
M_H in v, λ	N/A	$M_H^2 = 2\left\|\mu^2\right\| = 2v^2\lambda_{Kl}$	$M_H^2 = 2\left\|\mu^2\right\| = v^2\frac{\lambda_{Aitch}}{2} = 2v^2\lambda_{Kl}$	See (1.3) [3] Aitchison
Bottom line:	Because Aitchison uses $\lambda/4$ instead of λ, he needs to define ϕ_{min} differently to get the same v value. In summary, the two approaches define the μ^2 term in the Lagrangian with opposite signs and λ different by a factor of 4. The same $\left\|\mu\right\|$ and v are obtained in both, but one needs to exchange Aitchison's $\lambda/4$ for Klauber's λ to get all parameters like particle masses to agree in the two approaches. ϕ_{min} is not used to determine low energy parameters, as they are expressed in terms of μ, λ, and v, so one does not need to work with ϕ_{min}. But, ϕ_{min} differs in the two approaches by a factor of $\sqrt{2}$.			

In QFT, a real scalar mass squared term differs from a complex scalar by a factor of 2.

2.7 Appendix B

For RHS of (2-20),

$$\frac{1}{(\not{k}-m)^2} = \frac{(\not{k}+m)^2}{(\not{k}-m)^2(\not{k}+m)^2} = \frac{\not{k}^2+2\not{k}m+m^2}{\left((\not{k}-m)(\not{k}+m)\right)^2} = \frac{\not{k}^2+2\not{k}m+m^2}{\left(\not{k}^2-m^2\right)^2} = \frac{k^2+2\not{k}m+m^2}{\left(k^2-m^2\right)^2}. \tag{2-25}$$

The denominator is just a number, so only consider trace of numerator. Trace of a single gamma matrix is zero, so middle term in numerator drops out.

$$\int_0^\Lambda Tr\left(\frac{k^2+2\not{k}m+m^2}{\left(k^2-m^2\right)^2}\right)d^4k \sim \int_0^\Lambda Tr\left(\frac{k^2+m^2}{\left(k^2-m^2\right)^2}\right)k^3dk \xrightarrow[\text{energy}]{\text{high}} \sim \int_0^\Lambda Tr\left(\frac{1}{k^2}\right)k^3dk \sim \Lambda^2. \tag{2-26}$$

2.8 References

Aitchison, I. (2007). Supersymmetry in Particle Physics: An Elementary Introduction, Cambridge

Klauber, R. D. (2013). Student Friendly Quantum Field Theory Volume 1: Basic Principles and QED, Sandtrove

Klauber, R. D. (2021). Student Friendly Quantum Field Theory Volume 2: The Standard Model, Sandtrove

3 Infinitesimal Lorentz and Rotation Vector Transformations

3.1 Boost

$$ct' = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct - \tfrac{v}{c}x\right) \qquad x' = \frac{1}{\sqrt{1-v^2/c^2}}\left(x - \tfrac{v}{c}ct\right) \qquad y' = y \qquad z' = z \tag{3-1}$$

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} & & \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \xrightarrow{v<<c} \begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & -v/c & & \\ -v/c & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{3-2}$$

For $v/c << 1$

$$E' = E - vp^1 = E - v^1 p^1 \qquad p'^1 = p^1 - \frac{v}{c}\frac{E}{c} = p^1 - \frac{v^1}{c}\frac{E}{c} = p^1 - \frac{v^1}{c}\frac{mc^2}{c} = p^1 - mv^1 \qquad \frac{v^1}{c} << 1 \tag{3-3}$$

More generally,

$$E' = E - \mathbf{v}\mathbf{\cdot}\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - m\mathbf{v} \qquad \frac{v}{c} << 1 \tag{3-4}$$

For notation, sometimes we find $\boldsymbol{\eta} = (v^1, v^2, v^3)$ used. So, for low velocity,

$$E' = E - \boldsymbol{\eta}\mathbf{\cdot}\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - m\boldsymbol{\eta} \qquad \frac{v}{c} << 1 \tag{3-5}$$

For natural units, where $c = 1$, and $E = m$,

$$E' = E - \boldsymbol{\eta}\mathbf{\cdot}\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - \boldsymbol{\eta}E \qquad v << 1 \qquad \text{Aitchison (2.16) [20]} \tag{3-6}$$

3.2 Rotation

See Klauber, Vol. 2, (2-13), pg. 13 for the general rotation matrix, where the angles are successive ccw transformations about the axis labeled by the subscript on θ. (To be precise, θ_3 occurs first, θ_2, second, and θ_1 last.)

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & & & \\ & \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 & \sin\theta_2 \\ & \cos\theta_1\sin\theta_3 + \sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\cos\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3 & -\sin\theta_1\cos\theta_2 \\ & \sin\theta_1\sin\theta_3 - \cos\theta_1\sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{3-7}$$

For infinitesimal rotations, i.e., $\theta_i \to 0$, we have (Klauber, Vol. 2, (2-14), pg. 14)

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & -\theta_3 & \theta_2 \\ & \theta_3 & 1 & -\theta_1 \\ & -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \qquad \theta_i \ll 1\ . \tag{3-8}$$

$$E' = E \qquad p'^1 = -\theta_3 p^2 + \theta_2 p^3 \qquad p'^2 = \theta_3 p^1 - \theta_1 p^3 \qquad p'^3 = -\theta_2 p^1 + \theta_1 p^2\ . \tag{3-9}$$

The θ_i are really contravariant, so we should represent them by θ^i. Doing that we can write (3-9) as

$$p'^i = \varepsilon^{ijk}\theta^j p^k \quad \rightarrow \quad \mathbf{p}' = \boldsymbol{\theta}\times\mathbf{p} \qquad \theta^i << 1\ . \tag{3-10}$$

If we define a vector $\boldsymbol{\varepsilon} = (-\theta_1, -\theta_2, -\theta_3)$, then (3-9) and (3-10) become

$$E' = E \qquad \mathbf{p}' = -\boldsymbol{\varepsilon}\times\mathbf{p} \qquad \varepsilon^i << 1 \qquad \text{Aitchison (2.15) [20].} \tag{3-11}$$

4 Spinor Transforms in QFT

Page numbers and equation numbers with dashes in them below reference those in Klauber, *Student Friendly QFT Volume 1, Basic Principles and QED,* and *Volume 2, The Standard Model*. Be aware that notation is different from Aitchison, as explained earlier in this document. At the end of this Sect. 4, we will convert the result to Aitchison notation for use with that text.

4.1 Background

4.1.1 Weyl Representation Relations

In the Weyl rep, from Vol. 2, pg. 138,

$$_W\psi = \begin{bmatrix} _W\psi^L \\ _W\psi^R \end{bmatrix} = \begin{bmatrix} _W\psi_1 \\ _W\psi_2 \\ _W\psi_3 \\ _W\psi_4 \end{bmatrix}, \tag{4-1}$$

where the Weyl spinor is composed of a two-component left chiral (L superscript) spinor (also called an *L* Weyl spinor) and a two-component right chiral (R superscript) spinor (also called an *R* Weyl spinor).

Weyl rep gamma matrices, from vol. 2, pgs. 136 and 160, are

$$_W\gamma^0 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} \quad _W\gamma^1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \quad _W\gamma^2 = \begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix} \quad _W\gamma^3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}$$

$$_W\gamma^5 = \begin{bmatrix} -I & \\ & I \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{(5-10) and (5-88)}$$

4.1.2 Lorentz Transformation Group for Spinors

A general Lorentz transformation of spinors, including 3D rotations and boosts, is effected by the operator *D,* as shown in Vol. 2, pg. 153, LHS of (5-64) (and Vol. 1, pg. 171, (6-21))

$$\psi' = D\psi\ , \tag{4-2}$$

where, in Vol. 2, pg. 153, RHS of (5-64) (or Vol. 1 pg. 171, eq (6-22)), we stated without proof,

$$D = e^{-i\left(L^k\Theta^k + M^kQ^k\right)} \qquad \text{Klauber Vol 2 (5-64)}$$

$$L^k = -\tfrac{i}{2}\varepsilon_{ij}{}^k\gamma^i\gamma^j,\ \ \Theta^k = \underbrace{\left(\theta^1,\theta^2,\theta^3\right)}_{\substack{\text{rotation}\\ \text{angles}}},\ \ M^k = -\tfrac{i}{2}\gamma^0\gamma^k,\ \ Q^k = \underbrace{\underbrace{\left(v^1,v^2,v^3\right)}_{\text{boost velocity comps}}}_{\text{for } v \ll 1}\ . \qquad \text{Klauber Vol 2 (5-65)}$$

Note that (5-65) is for low velocities (and this is a correction listed in the corrections lists). For boost velocity a substantial fraction of the speed of light, Q^k takes a more complicated form. See the appendix herein.

4.2 Proofs in References

Relations (5-64) and (5-65) of Vol. 2 above are proven in the references of the footnote on the cited page of Vol. 1 (and also in S. Coleman, Quantum Field Theory, Lectures of Sidney Coleman, World Scientific 2019, pgs. 369-377), but they are done first for the LC spinor field in its own SU(2) space, then for the RC spinor fields in its separate SU(2) space. As can be seen by comparing the following with those derivations, (5-64) and (5-65) above express the same results in 4D spinor space instead of two separate 2D spaces.

4.3 3D Rotations

Dropping the *W* subscripts on the gamma matrices, but remembering we are working in the Weyl rep, we have, for a rotation only, without boost, from (5-64).

$$D = e^{-iL^k\Theta^k}\ , \tag{4-3}$$

where, from the LHS of Vol. 2 (5-65),

$$L^1=-\tfrac{i}{2}\left(\gamma^2\gamma^3-\gamma^3\gamma^2\right)\qquad L^2=-\tfrac{i}{2}\left(\gamma^3\gamma^1-\gamma^1\gamma^3\right)\qquad L^3=-\tfrac{i}{2}\left(\gamma^1\gamma^2-\gamma^1\gamma^2\right). \tag{4-4}$$

From Vol. 1, pg. 414, where the commutation relations for gamma matrices are shown, we have

$$\begin{gathered}L^3=-\tfrac{i}{2}\left(\gamma^1\gamma^2-\gamma^1\gamma^2\right)=-\begin{bmatrix}\sigma_3 & \\ & \sigma_3\end{bmatrix}=-\sigma_3\begin{bmatrix}1 & \\ & 1\end{bmatrix}\\ L^1=-\begin{bmatrix}\sigma_1 & \\ & \sigma_1\end{bmatrix}=-\sigma_1\begin{bmatrix}1 & \\ & 1\end{bmatrix}\qquad L^2=-\begin{bmatrix}\sigma_2 & \\ & \sigma_2\end{bmatrix}=-\sigma_2\begin{bmatrix}1 & \\ & 1\end{bmatrix},\end{gathered} \tag{4-5}$$

or, generally,

$$L^k=-\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}. \tag{4-6}$$

So, (4-3) becomes

$$D=e^{-iL^k\Theta^k}=e^{i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k} \tag{4-7}$$

For small rotations

$$D\psi=e^{i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k}\begin{bmatrix}\psi^L\\ \psi^R\end{bmatrix}\approx\left(I+i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k\right)\begin{bmatrix}\psi^L\\ \psi^R\end{bmatrix} \tag{4-8}$$

and the effect of a rotation on the *R* Weyl spinor is the same as that on the *L* Weyl spinor.

As noted, this effect is typically derived in texts separately on each of the two *SU*(2) spinors ψ^L and ψ^R, each in its own *SU*(2) space. Here, we have shown those relations in the full 4D spinor space of QFT.

4.4 Lorentz Boosts

From Vol. 2 (5-64) with boost, but no rotation,

$$D=e^{-\frac{1}{2}\gamma^0\gamma^k Q^k}. \tag{4-9}$$

where

$$\begin{gathered}\tfrac{1}{2}\gamma^0\gamma^1=\tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & 1\\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & 1\\ & & 1 & \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}-\sigma_1 & \\ & \sigma_1\end{bmatrix}\\ \tfrac{1}{2}\gamma^0\gamma^2=\tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & -i\\ & & i & \\ & i & & \\ -i & & & \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix} & i & & \\ -i & & & \\ & & & -i\\ & & i & \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}-\sigma_2 & \\ & \sigma_2\end{bmatrix}\\ \tfrac{1}{2}\gamma^0\gamma^3=\tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & -1\\ -1 & & & \\ & 1 & & \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}-\sigma_3 & \\ & \sigma_3\end{bmatrix}.\end{gathered} \tag{4-10}$$

Or, generally,

$$\tfrac{1}{2}\gamma^0\gamma^k=\tfrac{1}{2}\begin{bmatrix}-\sigma_k & \\ & \sigma_k\end{bmatrix}=-\tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}. \tag{4-11}$$

For a small boost, from Vol. 2 (5-65), (4-9), and (4-11),

$$D=e^{-\frac{1}{2}\gamma^0\gamma^k v^k}=e^{\frac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k}\approx I+\tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k\qquad v^k\ll 1. \tag{4-12}$$

so, from (4-1), (4-2), and (4-12),

$$D\psi=e^{-\frac{1}{2}\gamma^0\gamma^k v^k}\begin{bmatrix}\psi^L\\ \psi^R\end{bmatrix}\approx\left(I+\tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k\right)\begin{bmatrix}\psi^L\\ \psi^R\end{bmatrix}\qquad v^k\ll 1, \tag{4-13}$$

and the effect of a boost on the R Weyl spinor is the opposite of that on the L Weyl spinor.

As is the case for rotation, this effect is typically derived in texts separately on each of the two *SU*(2) spinors ψ^L and ψ^R, each in its own *SU*(2) space. Here, we have shown those relations in the full 4D spinor space of QFT.

4.5 Visualizing the Transformations

In this section, we show a heuristic visualization of a particle undergoing first a rotation and then, a boost. But, we should first note one thing about spins and momentum of RC (right chiral) and LC (left chiral) fermions.

4.5.1 Spin and Momentum of RC and LC Particles

From Vol. 2, pg. 139, (5-20), we know that the spin operator in the Weyl rep has form

$$\Sigma_3 = \tfrac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}. \tag{4-14}$$

So, the action of (4-14) on (4-1) is the same on the top two components (LC) of ψ, as on the bottom two components (RC).

$$\Sigma_3\,\psi = \tfrac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{4-15}$$

At speeds approaching the speed of light, chirality is the same as helicity. Consider, in that case, RC and LC particles to each have spin aligned with momentum **p**. From (4-15), they will have the same spin. But, at the speed of light, RC and LC particles are also RH (right hand helicity) and LH (left hand helicity) particles, so if they have the same spin, they must have opposite direction momentum.

We can generalize to any speed. That is, RC and LC particles with spin in the same direction have momentum in the opposite directions. We show this in Fig. 4-1 of the following section.

4.5.2 3D Rotation Visualization

In the top part of Fig. 4-1, we show RC (right chiral) and LC (left chiral) fermions at almost the speed of light, where 1) the spin virtually aligns with the momentum direction and pointing (RH rule) to the right (see Vol. 1, pgs. 95-96) and 2) helicity and chirality are essentially the same, i.e., the RC particle has RH (right hand helicity) and the LC particle has LH (left hand helicity).

In the lower part, LHS of the figure, we show RC and LC particles at speed much less than light, so spin does not align with momentum direction. The chirality of the particles remains the same, but the helicity changes (and the particles are no longer in helicity eigenstates).

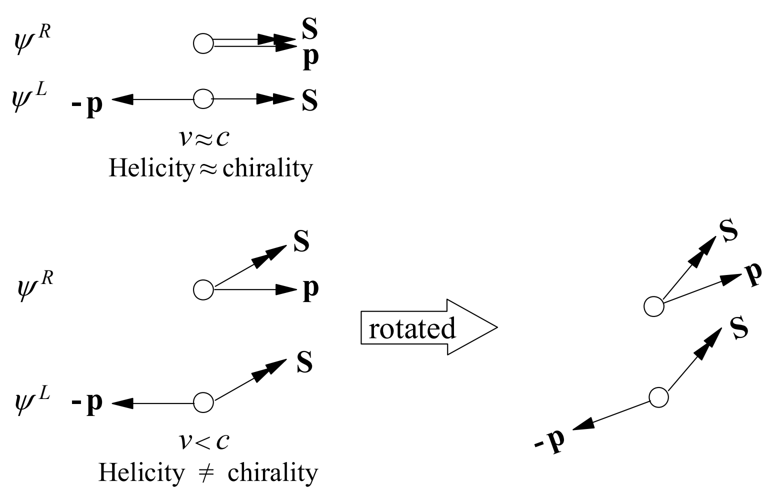


Figure 4-1. Heuristic Visualization of Fermion Rotation Transformation

In the lower part, RHS of the figure, we have rotated our reference frame. Note that both the RC and LC particles are affected in the same way, as we noted at the end of Sect. 4.3. The relationship between the spin and the momentum stays the same for both particles.

4.5.3 Boost Visualization

In Fig. 4-2, we show the same particles on the LHS as in the lower part LHS of Fig. 4-1. But note that when we boost both particles in the same direction, the momenta changes in opposite ways (one gets greater, the other lesser). Further, the spins change in opposite ways, as well. For an increase in momentum, spin gets closer to aligning with the direction of momentum. For a decrease in momentum, spin gets further from such alignment.

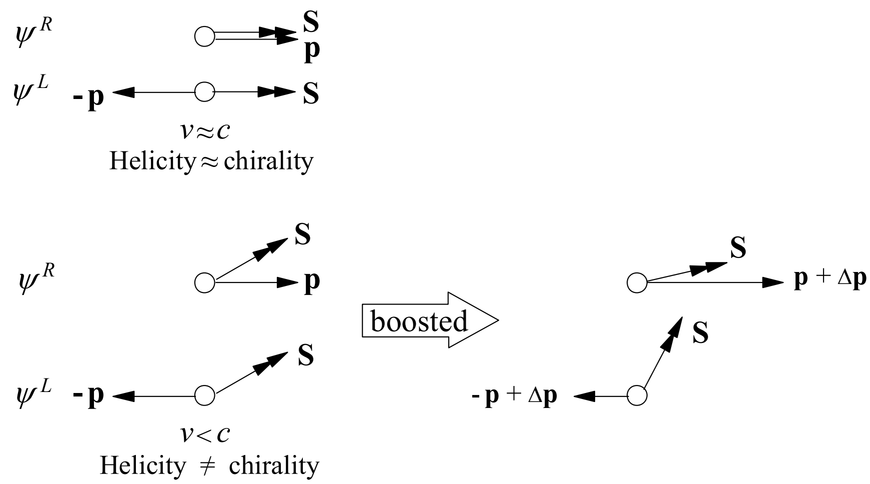


Figure 4-2. Heuristic Visualization of Fermion Boost Transformation

So, a boost transformation of an RC particle has the opposite effect of a boost on an LC particle, as we noted in Sect. 4.4.

4.6 Note on Derivation

Note that we have not derived the transformation of Vol. 2 (5-64). That is done in the references of the footnote of Vol. 1, pg. 171, and the derivations are long and complex. What we have done here is justify (5-64) of Vol. 2 to some degree in order to gain some intuitive level of comfort with the relation.

4.7 Summary and Notation Comparison

We can combine the small rotation transformation of (4-8) with the small Lorentz transformation of (4-13), and express the result in terms of the LC and RC Weyl fermions separately, as

$$\begin{bmatrix} \psi'^{L} \\ \psi'^{R} \end{bmatrix} \approx \begin{bmatrix} \left(I + i\sigma_k \theta^k + \frac{1}{2}\sigma_k v^k\right)\psi^L \\ \left(I + i\sigma_k \theta^k - \frac{1}{2}\sigma_k v^k\right)\psi^R \end{bmatrix} \qquad v^k \ll 1 \tag{4-16}$$

For Aitchison's notation (where he takes the LC field in the bottom position of the column vector and the RC field in the top position), we have

$$\psi^L \xrightarrow{\text{Aitchinson}} \chi \qquad \psi^R \xrightarrow{\text{Aitchinson}} \psi$$
$$\theta^k \xrightarrow{\text{Aitchinson}} \frac{\varepsilon}{2} \qquad v^k \xrightarrow{\text{Aitchinson}} \boldsymbol{\eta} \qquad (\text{bold} = \text{3-vector}) \tag{4-17}$$
$$\begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \xrightarrow{\text{Aitchinson}} \begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix} = \begin{bmatrix} \psi \\ \chi \end{bmatrix}$$

so, for Aitchison notation,

LC Weyl field $$\chi'_a = \left(1 + \underbrace{i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} + \underbrace{\boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\right)_a^{\ b} \chi_b = \left[V^{*-1}\right]_a^{\ b} \chi_b \qquad \text{infinitesimal, } a,b = 1,2 \tag{4-18}$$

simpler form $\rightarrow \chi' = V^{\dagger -1}\chi$

RC Weyl field $$\psi'^{\dot{a}} = \left(1 + \underbrace{i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} - \underbrace{\boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\right)^{\dot{a}}_{\ \dot{b}} \psi^{\dot{b}} = V^{\dot{a}}_{\ \dot{b}}\psi^{\dot{b}} \qquad \text{infinitesimal, } a,b = 1,2 \tag{4-19}$$

simpler form $\rightarrow \psi' = V\psi$

4.8 Appendix

The equivalent of Klauber Vol. 2 (5-65) for v^k a substantial fraction of the speed of light is more complicated. We start by assuming that in our coordinate system, x^3 is aligned with the direction of the boost **v**, and defining a parameter ϕ via

$$cosh\,\phi = \frac{1}{\sqrt{1-v^2}} \qquad sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \qquad tanh\,\phi = v\,. \tag{4-20}$$

Then, Q^k, which we state without proof (see earlier cited references), is

$$Q^k = (0,0,\phi)\,, \tag{4-21}$$

And Vol. 2 (5-64) is

$$D = e^{-i\left(L^k\Theta^k + M^kQ^k\right)} = e^{-i\left(L^k\Theta^k + M^3Q^3\right)} = e^{-i\left(L^k\Theta^k + M^3\phi\right)}. \tag{4-22}$$

This can be generalized to

$$D = e^{-i\left(L^k\Theta^k + M^k\phi^k\right)}, \tag{4-23}$$

where there are three ϕ^k for the general case of three components v^k.

In the limit of small v, from (4-20),

$$sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \xrightarrow{v \ll 1} sinh\,\phi \approx \phi \approx v\,, \tag{4-24}$$

And (4-22) becomes

$$D = e^{-i\left(L^k\Theta^k + M^3v\right)} = e^{-i\left(L^k\Theta^k\right)}e^{-\frac{1}{2}\gamma^0\gamma^3v^3}\,, \tag{4-25}$$

which we can generalize to Vol. 2 (5-64) with Vol. 2 (5-65) and (4-12) as

$$D = e^{-i\left(L^k\Theta^k\right)}e^{-\frac{1}{2}\gamma^0\gamma^kv^k} \qquad v^k \ll 1\,. \tag{4-26}$$

5 Spinor Space Notation, Metrics, and Transformations

5.1 Wholeness Chart 5-1. Metrics and Invariants in Different Spaces

Note: $\sigma_2 = \begin{bmatrix} & -i \\ i & \end{bmatrix}$ Also: using QFT spacetime metric form, not usual special relativity theory (SRT) form.

Math Entity	**3D Space, Cartesian**	**4D Spacetime, Minkowski, QFT**	**2D Spinor Space, Weyl Rep, LC**	**Comment**
Typical vector notation	x^i and x_i $\quad i,j = 1,2,3$	x^μ and x_μ $\quad \mu,\nu = 0,1,2,3$	χ_a and χ^a $\quad a,b = 1,2$	Not doing RC spinors yet.
Which is true vector?	x^i	x^μ	χ_a	LC covariant by convention
Which is calculation aid?	x_i	x_μ	χ^a	
General metric symbol	$g_{\mu\nu}$	as at left	as at left	
Specific metric symbol	$\delta_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $\delta^{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ $\eta^{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$	$\varepsilon^{ab} = i\sigma_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ $\varepsilon_{ab} = -i\sigma_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$	Each metric has form it does because it leads to invariant inner products. (See below.) For each, $g_{\mu\beta}\, g^{\beta\nu} = \delta_\mu{}^\nu$
Raising and lowering	$x_i = \delta_{ij} x^j \quad x^i = \delta^{ij} x_j$ $x^i = x_i$	$x_\mu = \eta_{\mu\nu} x^\nu \quad x^\mu = \eta^{\mu\nu} x_\nu$ $x_0 = x^0 \quad x_i = -x^i$	$\chi^a = \varepsilon^{ab} \chi_b \quad \chi_a = \varepsilon_{ab} \chi^b$ $\chi^1 = \chi_2 \quad \chi^2 = -\chi_1$	
Vector length squared	$x^i x_i = \delta_{ij} x^i x^j$	$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu$	$\chi_a \chi^a = \varepsilon^{ab} \chi_a \chi_b$	
Inner product, 2 vectors	$\underset{\sim}{x}^i x_i = \delta_{ij} \underset{\sim}{x}^i x^j$ $= \underset{\sim}{x}^1 x^1 + \underset{\sim}{x}^2 x^2 + \underset{\sim}{x}^3 x^3$ $= \underset{\sim}{x}_1 x_1 + \underset{\sim}{x}_2 x_2 + \underset{\sim}{x}_3 x_3$	$\underset{\sim}{x}^\mu x_\mu = \eta_{\mu\nu} \underset{\sim}{x}^\mu x^\nu$ $= \underset{\sim}{x}^0 x_0 + \underset{\sim}{x}^1 x_1 + \underset{\sim}{x}^2 x_2 + \underset{\sim}{x}^3 x_3$ $= \underset{\sim}{x}^0 x^0 - \underset{\sim}{x}^1 x^1 - \underset{\sim}{x}^2 x^2 - \underset{\sim}{x}^3 x^3$	$\underset{\sim}{\chi}^a \chi_a = \varepsilon_{ab} \underset{\sim}{\chi}^a \chi^b$ $= \underset{\sim}{\chi}^1 \chi_1 + \underset{\sim}{\chi}^2 \chi_2$ $= \underset{\sim}{\chi}_2 \chi_1 - \underset{\sim}{\chi}_1 \chi_2$	One can consider the negative of any inner product as invariant, as well.
Invariance	Above inner product invariant	Above inner product invariant	Above inner product invariant	Lorentz, rotation, and translation invariance, here
Where did above invariance come from?	Above is obvious.	We proved above in SRT course.	We haven't proven above yet.	

5.2 Two Component Spinor Notation: Weyl Rep

$$\sigma^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} & -i \\ i & \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \quad \sigma^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad \sigma^{\mu\dagger} = \sigma^{\mu} \tag{5-1}$$

$$\gamma^i = \begin{bmatrix} & -\sigma^i \\ \sigma^i & \end{bmatrix} \quad \gamma^0 = \begin{bmatrix} & I \\ I & \end{bmatrix} \quad \left(\gamma^{\mu} = \begin{bmatrix} & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & \end{bmatrix} \right) \qquad \gamma_5 = \begin{bmatrix} I & \\ & -I \end{bmatrix} \tag{5-2}$$

Weyl rep form is Aitchison's, not Klauber (or Schwartz or Peskin & Schroeder)

$$\text{Aitchison} \;\rightarrow\; \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \Psi^R \\ \Psi^L \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \quad \psi = \text{RC} \quad \chi = \text{LC} \qquad \text{Klauber, Schwartz, P\&S} \;\rightarrow\; \Psi = \begin{pmatrix} \Psi^L \\ \Psi^R \end{pmatrix} \tag{5-3}$$

Wholeness Chart 5-2. Comparing LC and RC Fields

Entity	**LC χ**	**RC ψ**	**Aitchison**
Default indices	lower $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$	upper $\psi = \begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix}$	convention
Index notation	not dotted, $\xi_a = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is LC	dotted, $\xi^{\dot a} = \begin{pmatrix} \xi^{\dot 1} \\ \xi^{\dot 2} \end{pmatrix}$ is RC	convention
Raised and lowered	$\chi^a = \varepsilon^{ab}\chi_b \quad \chi_a = \varepsilon_{ab}\chi^b$ Both LC, not dotted	$\psi_{\dot a} = \varepsilon_{\dot a\dot b}\psi^{\dot b} \quad \psi^{\dot a} = \varepsilon^{\dot a\dot b}\psi_{\dot b}$ Both RC, dotted	(2.61) [26] (2.71) [28]
Metric	$\varepsilon^{ab} = i\sigma_2^{ab} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \varepsilon^{\dot a\dot b}$	$\varepsilon_{\dot a\dot b} = \left(-i\sigma_2\right)_{ab} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \varepsilon_{ab}$	(2.63) [26] (2.72) [28]
χ, ψ notation	$\chi^a = \begin{bmatrix} \chi^1 \\ \chi^2 \end{bmatrix} = \varepsilon^{ab}\chi_b = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \chi_2 \\ -\chi_1 \end{bmatrix}$	$\psi_{\dot a} = \begin{bmatrix} \psi_{\dot 1} \\ \psi_{\dot 2} \end{bmatrix} = \varepsilon_{\dot a\dot b}\psi^{\dot b} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{bmatrix} = \begin{bmatrix} -\psi^{\dot 2} \\ \psi^{\dot 1} \end{bmatrix}$	(2.57) [25] (2.67) [27]
General notation	$\xi^a = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = \varepsilon^{ab}\xi_b = \begin{bmatrix} \xi_2 \\ -\xi_1 \end{bmatrix}$	$\xi_{\dot a} = \begin{bmatrix} \xi_{\dot 1} \\ \xi_{\dot 2} \end{bmatrix} = \varepsilon_{\dot a\dot b}\xi^{\dot b} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \xi^{\dot 1} \\ \xi^{\dot 2} \end{bmatrix} = \begin{bmatrix} -\xi^{\dot 2} \\ \xi^{\dot 1} \end{bmatrix}$	
Infinitesimal transformations	$\chi'_a = \Bigl(1 + \underbrace{i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} + \underbrace{\boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Bigr)_a^{\;b} \chi_b = \left[V^{*-1}\right]_a^{\;b}\chi_b$ simpler form $\rightarrow \chi' = V^{\dagger -1}\chi$	$\psi'^{\dot a} = \Bigl(1 + \underbrace{i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} - \underbrace{\boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Bigr)^{\dot a}_{\;\dot b} \psi^{\dot b} = V^{\dot a}_{\;\dot b}\psi^{\dot b}$ simpler form $\rightarrow \psi' = V\psi$	(2.24), (2.27), & (2.28) [21]
for raised/lowered	$\chi'^a = \left(1 - i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^a_{\;b} \chi^b = \left[V^*\right]^a_{\;b}\chi^b$	$\psi'_{\dot a} = \left(1 - i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)_{\dot a}^{\;\dot b} \psi_{\dot b} = \left[V^{-1}\right]_{\dot a}^{\;\dot b}\psi_{\dot b}$	Transform= above inverse transpose. Proof later.
Invariants	$\bar{\Psi}\Psi = \Psi^{\dagger}\gamma^0\Psi = \psi^{\dagger}\chi + \chi^{\dagger}\psi$ each of $\psi^{\dagger}\chi$ and $\chi^{\dagger}\psi$ independently invariant		(2.31) [22] (2.32) [22]
Inner products of spinor fields	$\underset{\sim}{\chi}\cdot\chi = \underset{\sim}{\chi}^a\chi_a = \left(\underset{\sim}{\chi}^1\ \underset{\sim}{\chi}^2\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \underset{\sim}{\chi}^1\chi_1 + \underset{\sim}{\chi}^2\chi_2$ $= \left(\underset{\sim}{\chi}_2\ -\underset{\sim}{\chi}_1\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \underset{\sim}{\chi}_2\chi_1 - \underset{\sim}{\chi}_1\chi_2 = -\underset{\sim}{\chi}^1\chi^2 + \underset{\sim}{\chi}^2\chi^1$ positive is top left to bottom right indices	$\underset{\sim}{\psi}\cdot\psi = \underset{\sim}{\psi}_{\dot a}\psi^{\dot a} = \left(\underset{\sim}{\psi}_{\dot 1}\ \underset{\sim}{\psi}_{\dot 2}\right)\begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix} = \underset{\sim}{\psi}_{\dot 1}\psi^{\dot 1} + \underset{\sim}{\psi}_{\dot 2}\psi^{\dot 2}$ $= \left(-\underset{\sim}{\psi}^{\dot 2}\ \underset{\sim}{\psi}^{\dot 1}\right)\begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix} = -\underset{\sim}{\psi}^{\dot 2}\psi^{\dot 1} + \underset{\sim}{\psi}^{\dot 1}\psi^{\dot 2} = \underset{\sim}{\psi}_{\dot 1}\psi_{\dot 2} - \underset{\sim}{\psi}_{\dot 2}\psi_{\dot 1}$ positive is bottom left to top right indices	Proof of invariance on next pages. (2.60) [26] (2.70) [28]
Covariants	$\bar{\Psi}\gamma^{\mu}\Psi = \psi^{\dagger}\sigma^{\mu}\chi + \chi^{\dagger}\bar{\sigma}^{\mu}\psi$ each of $\psi^{\dagger}\sigma^{\mu}\chi$ and $\chi^{\dagger}\bar{\sigma}^{\mu}\psi$ independently covariant $\sigma^{\mu} = \left(1, \sigma_i\right) \quad \bar{\sigma}^{\mu} = \left(1, -\sigma_i\right) \quad \gamma^{\mu} = \begin{bmatrix} & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & \end{bmatrix}$		(2.33) [22] (2.34) & (2.35) [22]
New notation		$\psi^{\dot 1} = \bar{\psi}^{\dot 1} \quad \psi^{\dot 2} = \bar{\psi}^{\dot 2} \rightarrow \underset{\sim}{\bar{\psi}}\cdot\bar{\psi} = \underset{\sim}{\psi}_{\dot a}\psi^{\dot a}$ Bar ≠ bar of QFT Dirac $\bar{\Psi}$	(2.76) [28]

5.3 Invariance of Weyl Spinor Inner Product Proof

5.3.1 Background

A very key relation, which is proven in the appendix of this Sect. 5 is this.

$$\sigma_2 \chi^* \text{ transforms like } \psi \quad \rightarrow \quad [\sigma_2]^{ab} \chi_b^* \text{ transforms like } \psi^a. \tag{5-4}$$

We will use this result for one step of the proof carried out in the next subsection.

5.3.2 Proof of Invariance of Weyl Spinor Inner Product

To prove the invariance of the Weyl spinor inner product $\chi^a \chi_a$, we proceed in numbered steps.

Step 1

We know from QFT that $\overline{\Psi}\Psi$ is a Lorentz, and rotation, invariant scalar.

$$\begin{aligned} \overline{\Psi}\Psi &= \begin{pmatrix} \psi \\ \chi \end{pmatrix}^\dagger \gamma^0 \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \psi^\dagger & \chi^\dagger \end{pmatrix} \begin{bmatrix} & I \\ I & \end{bmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \psi^\dagger \chi + \chi^\dagger \psi \\ &= \begin{pmatrix} \psi^{1*} & \psi^{2*} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \chi_1^* & \chi_2^* \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \psi^{a\dagger} \chi_a + \chi_a^\dagger \psi^a = \text{invariant} \end{aligned} \tag{5-5}$$

Step 2

It turns out that each of the terms in (5-5), $\psi^\dagger \chi$ and $\chi^\dagger \psi$ are invariant on their own, as we prove below.

For the first term, the factors transform (see Wholeness Chart 5-2) as

$$\chi' = V^{\dagger -1} \chi \qquad \psi' = V\psi \ \rightarrow \ \psi'^\dagger = \psi^\dagger V^\dagger. \tag{5-6}$$

So, $\psi^\dagger \chi$ is invariant via

$$\psi'^{a\dagger} \chi'_a = \psi'^\dagger \chi' = \left(\psi^\dagger V^\dagger\right)\left(V^{\dagger -1} \chi\right) = \psi^\dagger \left(V^\dagger V^{\dagger -1}\right) \chi = \psi^\dagger \chi = \psi^{a\dagger} \chi_a. \tag{5-7}$$

Homework Problem 5-1. Prove the invariance of the $\chi^\dagger \psi = \chi_a^\dagger \psi^a$ term, (It is rather straightforward.)

Step 3

We now combine the invariance result (5-7) with (5-4), but first note that since (5-4) is true, then, also,

$$i\sigma_2 \chi^* = \varepsilon^{ab} \chi_b^* = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} \chi_1^* \\ \chi_2^* \end{bmatrix} \text{ transforms like } \psi \ (= \psi^a). \tag{5-8}$$

And then,

$$-i\sigma_2^* \chi \text{ transforms like } \psi^*, \tag{5-9}$$

and, taking the transpose of (5-9),

$$\chi^T \left(-i\sigma_2^\dagger\right) = \chi^T \left(-i\sigma_2\right) \text{ transforms like } \psi^\dagger. \tag{5-10}$$

Step 4

So, for any $\psi^\dagger$ we might have, there exists a $\underset{\sim}{\chi}$ for which $\underset{\sim}{\chi}^T \left(-i\sigma_2\right)$ is equal to that $\psi^\dagger$. So, we can substitute (5-10) for $\psi^\dagger$ in (5-4). We'd actually like to use the symbol $\underset{\sim}{\chi}$ instead of χ in (5-10), so that $\underset{\sim}{\chi}$ represents any LC 2-spinor. (5-10) then gives us (5-11) below, which we know is invariant from (5-7),

$$\psi^\dagger \chi = \underset{\sim}{\chi}^T \left(-i\sigma_2\right) \chi = -\underset{\sim}{\chi}_a \varepsilon^{ab} \chi_b = -\begin{pmatrix} \underset{\sim}{\chi}_1 & \underset{\sim}{\chi}_2 \end{pmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = -\underset{\sim}{\chi}_1 \chi_2 + \underset{\sim}{\chi}_2 \chi_1 = \underset{\sim}{\chi}_2 \chi_1 - \underset{\sim}{\chi}_1 \chi_2 \text{ is invariant.} \tag{5-11}$$

<u>Step 5</u>

From Wholeness Chart 5-1 or 5-2,

$$\chi^a = \begin{bmatrix} \chi^1 \\ \chi^2 \end{bmatrix} = \varepsilon^{ab}\chi_b = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \chi_2 \\ -\chi_1 \end{bmatrix} \;\rightarrow\; \chi^1 = \chi_2 \quad \chi^2 = -\chi_1 \tag{5-12}$$

So,

$$\underset{\sim}{\chi}^a \chi_a = \left(\underset{\sim}{\chi}^1 \;\; \underset{\sim}{\chi}^2\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \left(\underset{\sim}{\chi}_2 \;\; -\underset{\sim}{\chi}_1\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \underset{\sim}{\chi}_2\chi_1 - \underset{\sim}{\chi}_1\chi_2 \,. \tag{5-13}$$

Comparing (5-13) to (5-11), we see that the inner product of two LC spinors is invariant and equals the last part of (5-13), as shown in Wholeness Chart 5-2.

For the inner product of a spinor with itself, the length squared in spinor space, (5-13) becomes

$$\chi^a \chi_a = \chi_2\chi_1 - \chi_1\chi_2 \,. \tag{5-14}$$

If χ were a number, (5-14) would be zero, but in QFT, χ is a field which anti-commutes with itself, so (5-14) is non-zero.

<u>First Note</u>

Consider

$$\underset{\sim}{\chi}_a \chi^a = \left(\underset{\sim}{\chi}_1 \;\; \underset{\sim}{\chi}_2\right)\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} = \left(\underset{\sim}{\chi}_1 \;\; \underset{\sim}{\chi}_2\right)\begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix} = \underset{\sim}{\chi}_1\chi_2 - \underset{\sim}{\chi}_2\chi_1 \,, \tag{5-15}$$

which is the negative of (5-13), though we would naively expect it to be equal to (5-13). This is an idiosyncrasy of the spinor space metric. Conventionally, (5-13) is considered the positive inner product of two LC spinors – index order from top left to bottom right.

<u>Second Note</u>

We know from (5-13) that $\underset{\sim}{\chi}^a \chi_a$ is invariant, and we know from Aitchison's (2.39) above, as well as Wholeness Chart 5-2, what the infinitesimal transformation is for an LC spinor like χ_a. So, we can deduce the transformation ($S^a{}_b$ here) for the contravariant form of a LC spinor $\underset{\sim}{\chi}^a$.

$$\underset{\sim}{\chi}'^a \chi'_a = \left[S^a{}_b \underset{\sim}{\chi}^b\right]\left[V^{\dagger-1}\right]_a{}^c \chi_c = S^a{}_b \left[V^{\dagger-1}\right]_a{}^c \underset{\sim}{\chi}^b \chi_c = S^a{}_b \left(1 + i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)_a{}^c \underset{\sim}{\chi}^b \chi_c \tag{5-16}$$

For

$$S^a{}_b = \left(1 - i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)^a{}_b \,, \tag{5-17}$$

(5-16) becomes

$$\begin{aligned}\underset{\sim}{\chi}'^a \chi'_a &= \left(1 - i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)^a{}_b \left(1 + i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)_a{}^c \underset{\sim}{\chi}^b \chi_c \\ &= \left(1 - i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} + i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2} + \textit{higher order in } \boldsymbol{\varepsilon} \textit{ and } \boldsymbol{\sigma}\right)_b{}^c \underset{\sim}{\chi}^b \chi_c .\end{aligned} \tag{5-18}$$

Since $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ are infinitesimal, the higher order terms are meaningless, so given (5-17),

$$\underset{\sim}{\chi}'^a \chi'_a = \delta^c_b \underset{\sim}{\chi}^b \chi_c = \underset{\sim}{\chi}^b \chi_b = \underset{\sim}{\chi}^a \chi_a \,, \tag{5-19}$$

i.e., the inner product is invariant, as we know it must be. Therefore, (5-17) is the correct transformation for the contravariant form of the LC spinor χ.

$$\underset{\sim}{\chi}'^{a} = \left(1 - i\varepsilon \bullet \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2}\right)^{a}_{\;b} \underset{\sim}{\chi}^{b} = \left[V^{*}\right]^{a}_{\;b} \underset{\sim}{\chi}^{b}, \tag{5-20}$$

as we show in Wholeness Chart 5-2.

5.4 Appendix

The following paragraphs are copied from Aitchison, pg. 23. For the (infinitesimal) Lorentz and rotation transformations for χ and ψ, referenced as (2.39) and (2.24) below, see Wholeness Chart 5-2 herein. Those transformation relations were derived in Sect. 4 herein, entitled Spinor Transforms in QFT.

Aitchison, pg. 23 copy:

complex conjugate of χ, denoted by χ^*, transforms under Lorentz transformations. We have

$$\chi' = (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\chi. \tag{2.39}$$

Taking the complex conjugate gives

$$\chi^{*\prime} = (1 - \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}^*/2 + \boldsymbol{\eta} \cdot \boldsymbol{\sigma}^*/2)\chi^*. \tag{2.40}$$

Now observe that $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$, and that $\sigma_2\sigma_3 = -\sigma_3\sigma_2$ and $\sigma_1\sigma_2 = -\sigma_2\sigma_1$. It follows that

$$\begin{aligned} \sigma_2\chi^{*\prime} &= \sigma_2(1 - \mathrm{i}\boldsymbol{\epsilon} \cdot (\sigma_1, -\sigma_2, \sigma_3)/2 + \boldsymbol{\eta} \cdot (\sigma_1, -\sigma_2, \sigma_3)/2)\chi^* &(2.41)\\ &= (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 - \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\sigma_2\chi^* &(2.42)\\ &= V\sigma_2\chi^*, &(2.43) \end{aligned}$$

referring to (2.24) for the definition of V, which is precisely the matrix by which ψ transforms.

We have therefore established the important result that

$$\sigma_2\chi^* \text{ transforms like a } \psi. \tag{2.44}$$

5.5 Solutions to Homework Problems

Homework Problem 5-1. Prove the invariance of the $\chi^\dagger\psi$ term, (It is rather straightforward.)

Ans. Using the transformations in Wholeness Chart 5-2, we have

$$\chi'^{\dagger}_{a}\psi'^{a} = (\chi')^{\dagger}\psi' = \left(V^{\dagger-1}\chi\right)^{\dagger} V\psi = \chi^{\dagger}\left(V^{\dagger\dagger}\right)^{-1} V\psi = \chi^{\dagger}V^{-1}V\psi = \chi^{\dagger}\psi = \chi^{\dagger}_{a}\psi^{a}. \tag{5-21}$$

6 Overview of How SUSY is Deduced

6.1 Steps to Deduce SUSY

The steps one uses to deduce supersymmetry parallel those used for the standard model U(1), SU(2), and SU(3). The following discussion of the steps involved in all these cases can be followed more easily by tracking them in Wholeness Chart 6-2 herein. The steps are

1. Propose a suitable Lagrangian density $\mathcal{L}$.
2. Find an internal symmetry of the action, typically represented by matrices operating on multiplets (column vectors of fields). Usually easiest to handle if the symmetry transformations are infinitesimal, i.e., each matrix operating on a column vector is multiplied by an arbitrary, real, small parameter, symbolized here by ε_i .
3. Note the commutation relations for the matrices that operate on the field multiplets. For $SU(n)$ theories, the matrices are generators of the Lie Algebra associated with the group transformation acting on the fields.
4. Use Noether's theorem to find the conserved 4-currents $j_i{}^{\mu}$.
5. Find the conserved charges Q_i (which will be operators) by integrating $j_i{}^0$ over all space.
6. Determine the commutation relations for the Q_i, which are the generators of the algebra for the group transformations acting on the states.
7. Use the commutation relations for Q_i to determine what effects each Q_i has when operating on particular states. (Sometimes they result in an eigenvalue [charge = quantum number] for a state. Sometimes they raise or lower a state, i.e., change it from a state with particular charge quantum number(s) to a state with different charge quantum number(s).)

Note: Aitchison does steps #6 and #7 differently (though he mentions in passing that #7can be done this way). See pgs. 53-57.

6.2 Simple SUSY Summary

When the early researchers did all of the above, they found, for the simplest form of SUSY, the following. Note particles are massless, so their helicity and chirality states are the same, i.e., LH = LC and RH= RC.

There are two charges, labeled Q_1 and Q_2. Q_1 changes an R spin (but LH, with **p** in opposite direction of R spin, i.e., **p** is in $-x^3$ direction) fermion into a scalar (ϕ herein). $Q_1{}^\dagger$ turns that scalar back into the original R spin (LH) fermion. Q_2 changes an L spin (also, LH with **p** in same direction as L spin, i.e., **p** is in $+x^3$ direction) fermion into a scalar. $Q_2{}^\dagger$ turns that scalar back into the original L spin, LH fermion.

Since the spin of a L spin fermion is –½, we say Q_2 "raises" that fermion to a zero spin scalar. Since the spin of a R spin fermion is +½, Q_1 is said to lower that fermion to a zero spin scalar. This is summarized in Wholeness Chart 6–1.

Wholeness Chart 6–1. The Effects of the Two SUSY Charges on States

Spin	Particle	Q_1	$Q_1{}^\dagger$	Q_2	$Q_2{}^\dagger$	Antiparticle	Q_1	$Q_1{}^\dagger$	Q_2	$Q_2{}^\dagger$
+ ½	LC fermion					RC antifermion				
		↓	↑						↑	↓
0	scalar ϕ					anti-scalar $\bar{\phi}$				
				↑	↓		↓	↑		
– ½	LC fermion					RC antifermion				

(Simlar effects for RC fermion ψ and LC anti-fermion $\psi^\dagger$)

$$\text{LC chiral (or super) multiplet} = \begin{bmatrix} \phi \\ \chi \end{bmatrix} \qquad \text{RC chiral (or super) multiplet} = \begin{bmatrix} \psi \\ \phi_R \end{bmatrix} \tag{6-1}$$

Note the chiral multiplets (or super multiplets) for the fields and the resulting operators on states.

$$\left[\delta_\varepsilon\right]\begin{bmatrix} 0 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 & \xi\bullet \\ -i\sigma^\mu \bar{\xi}\partial_\mu & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \chi \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix} \quad \left[\delta_\varepsilon^\dagger\right]\begin{bmatrix} \phi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \chi \end{bmatrix} \;\rightarrow\; Q_2\left|\chi\right\rangle = C\left|\phi\right\rangle \quad Q_2^\dagger\left|\phi\right\rangle = C'\left|\chi\right\rangle \tag{6-2}$$

Similar effects, though a bit more complicated, arise for spin 1 (gauge) fields/states and fermion fields/states.

Wholeness Chart 6-2. Operators on Fields vs States: Standard Model and SUSY

	QFT 4D Spin	SU(2) Isospin	SUSY	Comment
Field	$\psi = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_\mathbf{p}}}\left(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right)$ $u_1 = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\ 0\\ \frac{p^3}{E+m}\\ \frac{p^1+ip^2}{E+m}\end{pmatrix}$ etc. u_2, v_1, v_2	Isospin doublet $\Psi = \begin{pmatrix}\psi_u^L\\ \psi_d^L\end{pmatrix}$ or $\begin{pmatrix}\psi_{\nu_e}^L\\ \psi_e^L\end{pmatrix}$ generally $\begin{pmatrix}u^L\\ d^L\end{pmatrix}$ or $\Psi_a^L = \begin{pmatrix}\Psi_1^L\\ \Psi_2^L\end{pmatrix}$ u and d each like ψ at left and solves Dirac equation on its own; each is a four component (LC) spinor (indices not shown)	LC field, Weyl rep = χ $\Psi = \begin{bmatrix}\psi^{RC}\\ \psi^{LC}\end{bmatrix} = \begin{bmatrix}\psi\\ \chi\end{bmatrix}$ Ψ like ψ at far left or u or d at left. It could be an electron (or other fermion). Ψ has 4 components in spinor space. χ (and ψ) has 2 components	Column vector
Operator on fields	$\Sigma_i = \frac{\hbar}{2}\begin{bmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{bmatrix} \rightarrow \Sigma_1 = \frac{\hbar}{2}\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1\\ & & 1 & \end{bmatrix}$, $\Sigma_3 = \frac{\hbar}{2}\begin{bmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1\end{bmatrix}$ etc. for Σ_2	$\frac{1}{2}\tau_i = \frac{1}{2}\sigma_i \quad i=1,2,3$ Pauli matrices $SU(2)$ algebra generators for fields (See commutators below.) $\Psi'^L = \left(1 - i\varepsilon_i\tau_i/2\right)\Psi^L$ $\delta_{\varepsilon_i}\Psi^L = \begin{pmatrix}\delta_{\varepsilon_i}\psi_u^L\\ \delta_{\varepsilon_i}\psi_d^L\end{pmatrix} = -i\varepsilon_i\frac{\tau_i}{2}\begin{pmatrix}\psi_u^L\\ \psi_d^L\end{pmatrix}$	LC fermion & scalar doublet field $\begin{bmatrix}\phi\\ \chi\end{bmatrix}$ (chiral multiplet) $\delta_\xi\begin{bmatrix}\phi\\ \chi\end{bmatrix} = \begin{bmatrix} & \xi\bullet\\ -i\sigma^\mu\bar{\xi}\partial_\mu & \end{bmatrix}\begin{bmatrix}\phi\\ \chi\end{bmatrix}$ $\delta_\xi\phi = \xi\bullet\chi \quad \delta_\xi\chi = -i\sigma^\mu\bar{\xi}\partial_\mu\phi$	Matrix operator
Operator on states	${}_{\text{QFT}}\Sigma_i = \int_V \psi^\dagger\Sigma_i\psi\, d^3x$ $\rightarrow {}_{\text{QFT}}\Sigma_3 = \int_V \psi^\dagger\Sigma_3\psi\, d^3x$ $= \sum_{r,\mathbf{p}}\frac{m}{E_\mathbf{p}}\left(u_r^\dagger(\mathbf{p})\Sigma_3 u_r(\mathbf{p})N_r(\mathbf{p}) + v_r^\dagger(\mathbf{p})\Sigma_3 v_r(\mathbf{p})\bar{N}_r(\mathbf{p})\right)$	$T_i = \int_V j_i^0\, d^3x = \int_V \Psi^{L\dagger}\frac{\tau_i}{2}\Psi^L\, d^3x$ $\rightarrow T_3 = \frac{1}{2}\int_V \left(u^{L\dagger}\ d^{L\dagger}\right)\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{pmatrix}u^L\\ d^L\end{pmatrix}d$ $T_1 = \frac{1}{2}\int_V \left(u^{L\dagger}\ d^{L\dagger}\right)\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\begin{pmatrix}u^L\\ d^L\end{pmatrix}d^3x,$	$Q_a = \int_V J_a^0 d^3x \quad$ where $a=1,2$ $= \int_V \left(\sigma^\nu\chi(x)\right)_a\partial_\nu\phi^\dagger(x)d^3x$ Aitchison (4.71) [60]	From $\mathcal{L}$ symmetry → Noether conserved currents j^μ. Not a matrix.
Commu-tators	$\left[\Sigma_i,\Sigma_j\right] = i2\varepsilon_{ijk}\Sigma_k$ field operators $\left[{}_{\text{QFT}}\Sigma_i, {}_{\text{QFT}}\Sigma_j\right] = i2\varepsilon_{ijk}\,{}_{\text{QFT}}\Sigma_k$ state ops	$\left[\sigma_i,\sigma_j\right] = i2\varepsilon_{ijk}\sigma_k$ field opers $\left[T_i,T_j\right] = i2\varepsilon_{ijk}T_k$ state operators	Field opers not usually treated State operators: $\left[Q_a,Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab}P_\mu$ $\left[Q_a,P_\mu\right] = \left[Q_a^\dagger,P_\mu\right] = 0$	State com relations (found from above defs) not matrices.
State	for given r, $\mathbf{p}'$ $\lvert\psi_{r,\mathbf{p}'}\rangle$	for given d particle as LC e^-, $\lvert e^L_{r'\mathbf{p}'}\rangle$	for χ as LC e^-, $\lvert e^L_{r'\mathbf{p}'}\rangle$	Not a column vector
Operation on example state	For spin up state, ${}_{\text{QFT}}\Sigma_3\lvert\psi_{\uparrow,\mathbf{p}'}\rangle$ $= \frac{1}{2}\frac{m}{E_{\mathbf{p}'}}\underbrace{u_1^\dagger(\mathbf{p}')u_1(\mathbf{p}')}_{E_{\mathbf{p}'}/m}\lvert\psi_{\uparrow,\mathbf{p}'}\rangle$ $= \frac{1}{2}\lvert\psi_{\uparrow,\mathbf{p}'}\rangle$. spin = $+\frac{1}{2}$	For LC electron, $T_3\lvert e^L_{r'\mathbf{p}'}\rangle = \frac{1}{2}\left(N_{\nu r}(\mathbf{p}) - N_{er}(\mathbf{p})\right)\lvert e^L_{r'\mathbf{p}'}\rangle$ $= -\frac{1}{2}\lvert e^L_{r'\mathbf{p}'}\rangle \rightarrow$ weak charge $= -\frac{1}{2}$ $T_1\lvert e^L_{r'\mathbf{p}'}\rangle = A\lvert\nu^L_{e\,r'\mathbf{p}'}\rangle$ T_1 raises e^L to ν_e^L	$Q_2\lvert\chi\rangle = A\lvert\phi\rangle \quad A$ = a constant For LC electron, $Q_2\lvert e^L_{r'\mathbf{p}'}\rangle = A\underbrace{\lvert\tilde{e}^L_{r'\mathbf{p}'}\rangle}_{\text{selectron}}$ Q_2 raises χ to ϕ. $Q_2^\dagger$ reverse. Q_1 lowers χ to ϕ. $Q_1^\dagger$ reverse	Raising and lowering properties of state operators found using commutation relations.
Operation on states, in general	${}_{\text{QFT}}\Sigma_i\lvert\text{multipart}\rangle$ has no columns or matrices involved. Operator is just number, creation, destruction opers. For single particle up state, ${}_{\text{QFT}}\Sigma_3$ eigenvalue = ½, down state –½. ${}_{\text{QFT}}\Sigma_1$, ${}_{\text{QFT}}\Sigma_2$ raise & lower state spin.	$T_i\lvert e_{r,\mathbf{p}'}\rangle$ have no columns or matrices involved. Operator is just number, creation, destruction operators. For single electron state, T_3 eigenval = –½, neutrino = +½. T_1 & T_2 will exchange electron state with neutrino state.	$Q_a\lvert\text{state}\rangle$ have no columns or matrices involved. Just change fermion states to scalar states.	Terminology: creation = raising; destruction = lowering
Nomen-clature	${}_{\text{QFT}}\Sigma$ from diag Σ_3 is spin "charge". Some authors call all ${}_{\text{QFT}}\Sigma_i$ "charge".	T_i are generators of state algebra. T_3 from diag τ_3 is isospin charge. Some authors call all T_i "charge".	Q_a are called SUSY charges, or SUSY generators	

What we have looked at above is known as $N = 1$ SUSY. The "1" means our transformations do one thing – change the spin of a state by ½. As we will see shortly, SUSY theories for $N > 1$ have transformations that change the spin of a given particle by more than ½. For example, $N = 2$ SUSY can transform a particle spin by 1, in two steps of ½ each.

6.3 Clarification on Treating a Field Multiplet as a Vector or an Operator

In Aitchison (4.3) to (4.9), pg. 51, he considers an $SU(2)$ doublet of fields labeled q, where the u and d components represent (LC) fermion fields such as up and down quark fields, electron neutrino and electron fields, charm and strange quark fields, etc.

He then varies q in $SU(2)$ space. In (4.3) to (4.4), he treats varies q as if it were an entity in a vector space, i.e., the transformation is for a (2-component) vector. But, in (4.5) to (4.8), he treats q as if it were an operator that operates on a vector space, i.e., the transformation is for an operator, not a vector. That is, where equation numbers are those in Aitchison, the τ_i are the Pauli matrices, q transforms as a vector, the (unitary) transformation is infinitesimal (ε_i are infinitesimal real, arbitrary parameters), and herein we use the symbol $\hat{U}$ to represent that transformation,

$$q = \begin{bmatrix} u \\ d \end{bmatrix} \quad \text{(left chiral implied, not written in text)} \qquad \text{Aitchison (4.2) [51]} \qquad (6\text{-}3)$$

$$q' = q + \delta_\varepsilon q = \underbrace{\left(1 - i\boldsymbol{\varepsilon}\cdot\boldsymbol{\tau}/2\right)}_{\hat{U}} q = \left(1 - i\varepsilon_1 \tfrac{1}{2}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} - i\varepsilon_2 \tfrac{1}{2}\begin{bmatrix} & -i \\ i & \end{bmatrix} - i\varepsilon_3 \tfrac{1}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\right)\begin{bmatrix} u \\ d \end{bmatrix} = \hat{U}q\,. \qquad \text{Aitchison (4.3)} \qquad (6\text{-}4)$$

$$\delta_\varepsilon q = -i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\tau}}{2}q \qquad \text{Aitchison (4.4)} \qquad (6\text{-}5)$$

Normally, we think of the τ_i as the generators of infinitesimal transformations in the 2D vector space.

Aitchison follows this by transforming q as a matrix, or tensor, operator, where the (unitary) transformation is represented by U, and the T_i are generators of infinitesimal $SU(2)$ transformations.

$$q' = UqU^\dagger \qquad \text{Aitchison (4.5)} \qquad (6\text{-}6)$$

$$\begin{aligned} q' &= \left(1 + i\boldsymbol{\varepsilon}\cdot\boldsymbol{T}\right)q\left(1 - i\boldsymbol{\varepsilon}\cdot\boldsymbol{T}\right) = q + i\boldsymbol{\varepsilon}\cdot\boldsymbol{T}q - i\boldsymbol{\varepsilon}\cdot q\boldsymbol{T} + \textit{higher order terms} \\ &= q + i\boldsymbol{\varepsilon}\cdot\left[\boldsymbol{T}, q\right] \end{aligned} \qquad (6\text{-}7)$$

$$\delta_\varepsilon q = -i\boldsymbol{\varepsilon}\cdot\left[\boldsymbol{T}, q\right] \qquad \text{part of Aitchison (4.9)} \qquad (6\text{-}8)$$

Aitchison (4.4) is deduced considering q as a vector, but his (4.9) is deduced considering q as an operator (typically operating on a vector). One might ask how can one do such a thing. q must be either a vector or an operator, no?

The answer to this question lies in the overall structure of QFT. Texts rarely note there are really two vector spaces involved.

One such vector space is the vector space of fields, represented in QFT, as, for examples, u as an up quark field, d as a down quark field, or ν_e as an electron neutrino field. These fields create and destroy states. In $SU(2)$ space, for electroweak interactions, the left chiral (LC) parts of these fields are configured into doublets, i.e., 2 component vectors, such as in Aitchison (4.2). (The u and d therein are actually LC, as the RC u and d fields are $SU(2)$ singlets (think "scalars").

A different (but related via QFT) vector space is the vector space of states, represented in QFT, as, for examples, $|u\rangle$ as an up quark, $|d\rangle$ as a down quark, or $|\nu_e\rangle$ as an electron neutrino. In this context, the u up quark field is an operator that creates and destroys up quark states (particles). Similarly, d and ν_e and other fields create and destroy other types of particle states.

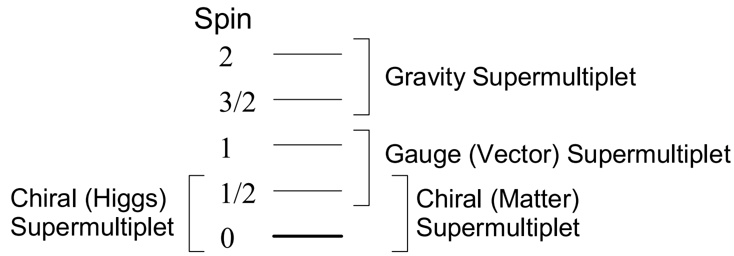
Wholeness Chart 6-2 lays out the differences between these two vector spaces. For a pedagogic overview, see also http://www.quantumfieldtheory.info/Opers_Fields_States.pdf.

Bottom line: In the transformation of Aitchison (4.3) and (4.4), one considers q to be a vector in the $SU(2)$ space of fields. In the transformation of Aitchison (4.5) and (4.9), one considers each component of q to be an operator that operates on the vector space of particle states. Whereas τ_i in Aitchison (4.4) are matrices, the T_i in (4.9) are not. The T_i are composed of creation, destruction, and number operators, and those result in commutation relations between the T_i, which are considered to define the Lie Algebra of transformations in state space. The particle states are not column vectors *per se*, but simply comprise one or more particles.

6.4 *N* = 1 SUSY: Wholeness Chart Overviews

6.4.1 Supermultiplets

See Wholeness Chart 6-3 for a simple overview of the supermultiplets (superdoublets, superfields) for N=1 SUSY.



Wholeness Chart 6-3. Supersymmetric Multiplets for N=1 Supersymmetry

See Wholeness Charts 6-5 to 6-7 for a summary of the action of the operators in the simplest form of SUSY, $N = 1$, and a comparison of multiplets in superspace with multiplets in $SU(2)$ weak interaction space. Note that in the wholeness charts, we are considering all massless particles (before Higgs symmetry breaking), so chirality and helicity for fermions are the same thing. RC = RH, and LC = LH.

6.4.2 Operator Anti-commutators/Commutators

For $N = 1$ SUSY, we have one operator, Q_1, that lowers spin +½ LH fermions to (helicity zero) scalars and another, Q_2, that raises spin − ½ LH fermions to scalars. As we will prove, these obey the following commutator/anticommutator relations, for which $a, b = 1,2$,

$$
\begin{aligned}
&\left[Q_a, Q_b^\dagger\right]_+ = (\sigma^\mu)_{ab} P_\mu \qquad \left(= I_{ab}P_0 + \sigma^1_{ab}P_1 + \sigma^2_{ab}P_2 + \sigma^3_{ab}P_3\right) \\
&\left[Q_a, Q_b\right]_+ = 0 \\
&\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0.
\end{aligned}
\tag{6-9}
$$

6.5 N > 1 SUSY: A Simple Look

6.5.1 N=2 SUSY

For $N = 2$, we would find, where superscripts are labels (not powers), and Z^{12}, known as the central charge, can be derived,

$$
\begin{aligned}
&\left[Q_a^1, Q_b^{1\,\dagger}\right]_+ = (\sigma^\mu)_{ab} P_\mu \qquad \left[Q_a^2, Q_b^{2\,\dagger}\right]_+ = (\sigma^\mu)_{ab} P_\mu \qquad \left[Q_a^1, Q_b^{2\,\dagger}\right]_+ = \left[Q_a^2, Q_b^{1\,\dagger}\right]_+ = 0 \\
&\left[Q_a^1, Q_b^2\right]_+ = \left[Q_a^2, Q_b^1\right]_+ = Z^{12} \qquad \left[Q_a^1, Q_b^1\right]_+ = \left[Q_a^2, Q_b^2\right]_+ = 0 \\
&\left[Q_a^1, P_\mu\right] = \left[Q_a^{1\dagger}, P_\mu\right] = \left[Q_a^2, P_\mu\right] = \left[Q_a^{2\dagger}, P_\mu\right] = 0
\end{aligned}
\tag{6-10}
$$

for which $a, b = 1,2$, again, so there are four operators (plus their complex conjugate transposes). Alternatively, we can label the four operators as Q_a, with $a = 1,2,3,4$.

For $N = 2$, a first operator lowers a +½ spin fermion to a scalar. Then a second operator lowers that scalar to a spin −½ fermion. Then, a third operator raises that to a second scalar. Finally, a fourth operator raises that to a +½ fermion. The net result is a quartet of the four fields, rather than a doublet. See Wholeness Charts 6-4 and 6-8, where the reverse transformations $Q_i^\dagger$ are not shown.

Wholeness Chart 6-4. The Effects of $N = 2$ SUSY Operators on States

Spin	Particle	Q_1	Q_2	Q_3	Q_4
+ ½	RC fermion	ψ			ψ
		$\downarrow$			$\uparrow$
0	scalars	ϕ	ϕ	$\tilde{\phi}^\dagger$	$\tilde{\phi}^\dagger$
			$\downarrow$	$\uparrow$	
– ½	LC fermion		χ	χ	

($Q^\dagger_a$ do the reverse of Q_a)

Since SUSY transformations cannot change any $SU(3)$, $SU(2)$, or $U(1)$ charges, $N = 2$ SUSY gives us an LC fermion with the same SM interactions behavior as an RC fermion. But, we know that is not true for $SU(2)$ (weak) interactions. So, at least in its simplest and most direct form, SUSY cannot be an adjunct to the standard model unless $N = 1$.

6.6 SUSY Supermultiplets Overview

Wholeness Chart 6-5. N= 1 Matter (j = ½) Supermultiplets

Left Chiral Superdoublets

(|0⟩ here means zero spin state, not vacuum state)

	Superspace		SUSY Operator	SU(2) Space	
	States	Fields		States	Fields
Particles (LC= LH) & sparticles	$\lvert 0\rangle_L$ & $\lvert -\tfrac{1}{2}\rangle_L$ or $\lvert \phi_\chi\rangle$ & $\lvert \chi\rangle$	Doublets $\begin{pmatrix}\phi_\chi\\ \chi\end{pmatrix}$	$Q_2\lvert -\tfrac{1}{2}\rangle_L = A\lvert 0\rangle_L \quad Q_2^\dagger\lvert 0\rangle_L = B\lvert -\tfrac{1}{2}\rangle_L$ or $Q_2\lvert \chi\rangle = A\lvert \phi_\chi\rangle \quad Q_2^\dagger\lvert \phi_\chi\rangle = B\lvert \chi\rangle$	$\lvert -\tfrac{1}{2}\rangle_L \overset{\text{SUSY}}{\Leftrightarrow} \lvert 0\rangle_L$ or $\lvert \chi\rangle \overset{\text{SUSY}}{\Leftrightarrow} \lvert \phi_\chi\rangle$	Doublets $\begin{pmatrix}\chi_u\\ \chi_d\end{pmatrix} \overset{\text{SUSY}}{\Leftrightarrow} \begin{pmatrix}\tilde{\chi}_u\\ \tilde{\chi}_d\end{pmatrix}$
Examples	$\lvert \tilde{e}_L\rangle$ & $\lvert e_L\rangle$ $\lvert \tilde{\nu}_{e_L}\rangle$ & $\lvert \nu_{e_L}\rangle$ $\lvert \tilde{u}_L\rangle$ & $\lvert u_L\rangle$ $\lvert \tilde{d}_L\rangle$ & $\lvert d_L\rangle$	$\begin{pmatrix}\tilde{e}_L\\ e_L\end{pmatrix}, \begin{pmatrix}\tilde{\nu}_{e_L}\\ \nu_{e_L}\end{pmatrix},$ $\begin{pmatrix}\tilde{u}_L\\ u_L\end{pmatrix}, \begin{pmatrix}\tilde{d}_L\\ d_L\end{pmatrix}$	$Q_2\lvert e_L\rangle = A\lvert \tilde{e}_L\rangle \quad Q_2^\dagger\lvert \tilde{e}_L\rangle = B\lvert e_L\rangle$ $Q_2\lvert \nu_{e_L}\rangle = A\lvert \tilde{\nu}_{e_L}\rangle \quad Q_2^\dagger\lvert \tilde{\nu}_{e_L}\rangle = B\lvert \nu_{e_L}\rangle$ $Q_2\lvert u_L\rangle = A\lvert \tilde{u}_L\rangle \quad Q_2^\dagger\lvert \tilde{u}_L\rangle = B\lvert u_L\rangle$ $Q_2\lvert d_L\rangle = A\lvert \tilde{d}_L\rangle \quad Q_2^\dagger\lvert \tilde{d}_L\rangle = B\lvert d_L\rangle$	$\lvert e_L\rangle \Leftrightarrow \lvert \tilde{e}_L\rangle$ $\lvert \nu_{e_L}\rangle \Leftrightarrow \lvert \tilde{\nu}_{e_L}\rangle$ $\lvert u_L\rangle \Leftrightarrow \lvert \tilde{u}_L\rangle$ $\lvert d_L\rangle \Leftrightarrow \lvert \tilde{d}_L\rangle$	$\begin{pmatrix}\nu_{e_L}\\ e_L\end{pmatrix} \Leftrightarrow \begin{pmatrix}\tilde{\nu}_{e_L}\\ \tilde{e}_L\end{pmatrix}$ $\begin{pmatrix}u_L\\ d_L\end{pmatrix} \Leftrightarrow \begin{pmatrix}\tilde{u}_L\\ \tilde{d}_L\end{pmatrix}$
Antiparticles (RC) and anti-sparticles	$\lvert +\tfrac{1}{2}\rangle_{\bar{R}}$ & $\lvert 0\rangle_{\bar{R}}$ or $\lvert \bar{\psi}\rangle$ & $\lvert \phi_{\bar{\psi}}\rangle$	$\begin{pmatrix}\bar{\psi}\\ \phi_{\bar{\psi}}\end{pmatrix}$	$Q_2^\dagger\lvert +\tfrac{1}{2}\rangle_{\bar{R}} = A\lvert 0\rangle_{\bar{R}} \quad Q_2\lvert 0\rangle_{\bar{R}} = B\lvert +\tfrac{1}{2}\rangle_{\bar{R}}$ or $Q_2^\dagger\lvert \bar{\psi}\rangle = A\lvert \phi_{\bar{\psi}}\rangle \quad Q_2\lvert \phi_{\bar{\psi}}\rangle = B\lvert \bar{\psi}\rangle$	$\lvert +\tfrac{1}{2}\rangle_{\bar{R}} \Leftrightarrow \lvert 0\rangle_{\bar{R}}$ or $\lvert \bar{\psi}\rangle \Leftrightarrow \lvert \phi_{\bar{\psi}}\rangle$	Adjoint doublets in rows $(\bar{\psi}_d \;\; \bar{\psi}_u) \Leftrightarrow (\tilde{\bar{\psi}}_d \;\; \tilde{\bar{\psi}}_u)$
Examples	$\lvert \tilde{\bar{e}}_R\rangle$ & $\lvert \bar{e}_R\rangle$ $\lvert \tilde{\bar{\nu}}_{e_R}\rangle$ & $\lvert \bar{\nu}_{e_R}\rangle$ $\lvert \tilde{\bar{u}}_R\rangle$ & $\lvert \bar{u}_R\rangle$ $\lvert \tilde{\bar{d}}_R\rangle$ & $\lvert \bar{d}_R\rangle$	$\begin{pmatrix}\bar{e}_R\\ \tilde{\bar{e}}_R\end{pmatrix}, \begin{pmatrix}\nu_{e_L}\\ \tilde{\bar{\nu}}_{e_R}\end{pmatrix},$ $\begin{pmatrix}\bar{u}_R\\ \tilde{\bar{u}}_R\end{pmatrix}, \begin{pmatrix}\bar{d}_R\\ \tilde{\bar{d}}_R\end{pmatrix}$	$Q_2^\dagger\lvert \bar{e}_R\rangle = A\lvert \tilde{\bar{e}}_R\rangle \quad Q_2\lvert \tilde{\bar{e}}_R\rangle = B\lvert \bar{e}_R\rangle$ $Q_2^\dagger\lvert \bar{\nu}_{e_R}\rangle = A\lvert \tilde{\bar{\nu}}_{e_R}\rangle \quad Q_2\lvert \bar{\nu}_{e_R}\rangle = B\lvert \tilde{\bar{\nu}}_{e_R}\rangle$ $Q_2^\dagger\lvert \bar{u}_R\rangle = A\lvert \tilde{\bar{u}}_R\rangle \quad Q_2\lvert \tilde{\bar{u}}_R\rangle = B\lvert \bar{u}_R\rangle$ $Q_2^\dagger\lvert \bar{d}_R\rangle = A\lvert \tilde{\bar{d}}_R\rangle \quad Q_2\lvert \tilde{\bar{d}}_R\rangle = B\lvert \bar{d}_R\rangle$	$\lvert \bar{e}_R\rangle \Leftrightarrow \lvert \tilde{\bar{e}}_R\rangle$ $\lvert \bar{\nu}_{e_R}\rangle \Leftrightarrow \lvert \tilde{\bar{\nu}}_{e_R}\rangle$ $\lvert \bar{u}_R\rangle \Leftrightarrow \lvert \tilde{\bar{u}}_R\rangle$ $\lvert \bar{d}_R\rangle \Leftrightarrow \lvert \tilde{\bar{d}}_R\rangle$	$(\bar{e}_R \;\; \bar{\nu}_{e_R}) \Leftrightarrow (\tilde{\bar{e}}_R \;\; \tilde{\bar{\nu}}_{e_R})$ $(\bar{d}_R \;\; \bar{u}_R) \Leftrightarrow (\tilde{\bar{d}}_R \;\; \tilde{\bar{u}}_R)$

Right Chiral Superdoublets

	Superspace		SUSY Operator	SU(2) Space	
	States	Fields		States	Fields
Particles (RC) and sparticles	$\lvert 0\rangle_R$ & $\lvert +\tfrac{1}{2}\rangle_R$ or $\lvert \phi_\psi\rangle$ & $\lvert \psi\rangle$	Doublets $\begin{pmatrix}\psi\\ \phi_\psi\end{pmatrix}$	$Q_1\lvert +\tfrac{1}{2}\rangle_R = A\lvert 0\rangle_R \quad Q_1^\dagger\lvert 0\rangle_R = B\lvert +\tfrac{1}{2}\rangle_R$ or $Q_1\lvert \psi\rangle = A\lvert \phi_\psi\rangle \quad Q_1^\dagger\lvert \phi_\psi\rangle = B\lvert \psi\rangle$	$\lvert +\tfrac{1}{2}\rangle_R \Leftrightarrow \lvert 0\rangle_R$ or $\lvert \psi\rangle \Leftrightarrow \lvert \phi_\psi\rangle$	Singlets $\psi_u \Leftrightarrow \tilde{\psi}_u$ $\psi_d \Leftrightarrow \tilde{\psi}_d$
Examples	$\lvert \tilde{e}_R\rangle$ & $\lvert e_R\rangle$ $\lvert \tilde{\nu}_{e_R}\rangle$ & $\lvert \nu_{e_R}\rangle$ $\lvert \tilde{u}_R\rangle$ & $\lvert u_R\rangle$ $\lvert \tilde{d}_R\rangle$ & $\lvert d_R\rangle$	$\begin{pmatrix}e_R\\ \tilde{e}_R\end{pmatrix}, \begin{pmatrix}\nu_{e_R}\\ \tilde{\nu}_{e_R}\end{pmatrix},$ $\begin{pmatrix}u_R\\ \tilde{u}_R\end{pmatrix}, \begin{pmatrix}d_R\\ \tilde{d}_R\end{pmatrix}$	$Q_1\lvert e_R\rangle = A\lvert \tilde{e}_R\rangle \quad Q_1^\dagger\lvert \tilde{e}_R\rangle = B\lvert e_R\rangle$ $Q_1\lvert \nu_{e_R}\rangle = A\lvert \tilde{\nu}_{e_R}\rangle \quad Q_1^\dagger\lvert \nu_{e_R}\rangle = B\lvert \tilde{\nu}_{e_R}\rangle$ $Q_1\lvert u_R\rangle = A\lvert \tilde{u}_R\rangle \quad Q_1^\dagger\lvert \tilde{u}_R\rangle = B\lvert u_R\rangle$ $Q_1\lvert d_R\rangle = A\lvert \tilde{d}_R\rangle \quad Q_1^\dagger\lvert \tilde{d}_R\rangle = B\lvert d_R\rangle$	$\lvert e_R\rangle \Leftrightarrow \lvert \tilde{e}_R\rangle$ $\lvert \nu_{e_R}\rangle \Leftrightarrow \lvert \tilde{\nu}_{e_R}\rangle$ $\lvert u_R\rangle \Leftrightarrow \lvert \tilde{u}_R\rangle$ $\lvert d_R\rangle \Leftrightarrow \lvert \tilde{d}_R\rangle$	$\nu_{e_R} \Leftrightarrow \tilde{\nu}_{e_R}$ $e_R \Leftrightarrow \tilde{e}_R$ $u_R \Leftrightarrow \tilde{u}_R$ $d_R \Leftrightarrow \tilde{d}_R$
Antiparticles (LC) and anti-sparticles	$\lvert -\tfrac{1}{2}\rangle_{\bar{L}}$ & $\lvert 0\rangle_{\bar{L}}$ or $\lvert \phi_{\bar{\chi}}\rangle$ & $\lvert \bar{\chi}\rangle$	$\begin{pmatrix}\bar{\chi}\\ \phi_{\bar{\chi}}\end{pmatrix}$	$Q_1^\dagger\lvert -\tfrac{1}{2}\rangle_{\bar{L}} = A\lvert 0\rangle_{\bar{L}} \quad Q_1\lvert 0\rangle_{\bar{L}} = B\lvert -\tfrac{1}{2}\rangle_{\bar{L}}$ or $Q_1^\dagger\lvert \bar{\chi}\rangle = A\lvert \phi_{\bar{\chi}}\rangle \quad Q_1\lvert \phi_{\bar{\chi}}\rangle = B\lvert \bar{\chi}\rangle$	$\lvert -\tfrac{1}{2}\rangle_{\bar{L}} \Leftrightarrow \lvert 0\rangle_{\bar{L}}$ or $\lvert \bar{\chi}\rangle \Leftrightarrow \lvert \phi_{\bar{\chi}}\rangle$	$\bar{\chi}_u \Leftrightarrow \tilde{\bar{\chi}}_u$ $\bar{\chi}_d \Leftrightarrow \tilde{\bar{\chi}}_d$
Examples	$\lvert \tilde{\bar{e}}_R\rangle$ & $\lvert \bar{e}_R\rangle$ $\lvert \tilde{\bar{\nu}}_{e_R}\rangle$ & $\lvert \bar{\nu}_{e_R}\rangle$ $\lvert \tilde{\bar{u}}_R\rangle$ & $\lvert \bar{u}_R\rangle$ $\lvert \tilde{\bar{d}}_R\rangle$ & $\lvert \bar{d}_R\rangle$	$\begin{pmatrix}\bar{e}_R\\ \tilde{\bar{e}}_R\end{pmatrix}, \begin{pmatrix}\nu_{e_L}\\ \tilde{\bar{\nu}}_{e_R}\end{pmatrix},$ $\begin{pmatrix}\bar{u}_R\\ \tilde{\bar{u}}_R\end{pmatrix}, \begin{pmatrix}\bar{d}_R\\ \tilde{\bar{d}}_R\end{pmatrix}$	$Q_1^\dagger\lvert \bar{e}_R\rangle = A\lvert \tilde{\bar{e}}_R\rangle \quad Q_1\lvert \tilde{\bar{e}}_R\rangle = B\lvert \bar{e}_R\rangle$ $Q_1^\dagger\lvert \bar{\nu}_{e_R}\rangle = A\lvert \tilde{\bar{\nu}}_{e_R}\rangle \quad Q_1\lvert \bar{\nu}_{e_R}\rangle = B\lvert \tilde{\bar{\nu}}_{e_R}\rangle$ $Q_1^\dagger\lvert \bar{u}_R\rangle = A\lvert \tilde{\bar{u}}_R\rangle \quad Q_1\lvert \tilde{\bar{u}}_R\rangle = B\lvert \bar{u}_R\rangle$ $Q_1^\dagger\lvert \bar{d}_R\rangle = A\lvert \tilde{\bar{d}}_R\rangle \quad Q_1\lvert \tilde{\bar{d}}_R\rangle = B\lvert \bar{d}_R\rangle$	$\lvert \bar{e}_R\rangle \Leftrightarrow \lvert \tilde{\bar{e}}_R\rangle$ $\lvert \bar{\nu}_{e_R}\rangle \Leftrightarrow \lvert \tilde{\bar{\nu}}_{e_R}\rangle$ $\lvert \bar{u}_R\rangle \Leftrightarrow \lvert \tilde{\bar{u}}_R\rangle$ $\lvert \bar{d}_R\rangle \Leftrightarrow \lvert \tilde{\bar{d}}_R\rangle$	$\bar{e}_R \Leftrightarrow \tilde{\bar{e}}_R$ $\bar{\nu}_{e_R} \Leftrightarrow \tilde{\bar{\nu}}_{e_R}$ $\bar{u}_R \Leftrightarrow \tilde{\bar{u}}_R$ $\bar{d}_R \Leftrightarrow \tilde{\bar{d}}_R$

Wholeness Chart 6-6. N= 1 Gauge (j = 1), or Vector, Superdoublets

Left Helicity Vector Bosons

	Superspace		SUSY Operator	SU(2) Space	
	States	Fields		States	Fields
Particles (LH) and sparticles (LH=LC)	$\lvert W^+\rangle$ & $\lvert\tilde{W}^+\rangle$ $\lvert Z\rangle$ & $\lvert\tilde{Z}\rangle$ $\lvert\gamma\rangle$ & $\lvert\tilde{\gamma}\rangle$ $\lvert g\rangle$ & $\lvert\tilde{g}\rangle$	Doublets $\begin{pmatrix}\tilde{W}^+\\ W^+\end{pmatrix},\begin{pmatrix}\tilde{Z}\\ Z\end{pmatrix}$ $\begin{pmatrix}\tilde{\gamma}\\ \gamma\end{pmatrix},\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}$	$Q_2'\lvert W^+\rangle=A'\lvert\tilde{W}^+\rangle$ $Q_2'^{\dagger}\lvert\tilde{W}^+\rangle=B'\lvert W^+\rangle$ $Q_2'\lvert Z\rangle=A'\lvert\tilde{Z}\rangle$ $Q_2'^{\dagger}\lvert\tilde{Z}\rangle=B'\lvert Z\rangle$ $Q_2'\lvert\gamma\rangle=A'\lvert\tilde{\gamma}\rangle$ $Q_2'^{\dagger}\lvert\tilde{\gamma}\rangle=B'\lvert\gamma\rangle$ $Q_2'\lvert g\rangle=A'\lvert\tilde{g}\rangle$ $Q_2'^{\dagger}\lvert\tilde{g}\rangle=B'\lvert g\rangle$	$\lvert W^+\rangle\Leftrightarrow\lvert\tilde{W}^+\rangle$ $\lvert Z\rangle\Leftrightarrow\lvert\tilde{Z}\rangle$ $\lvert\gamma\rangle\Leftrightarrow\lvert\tilde{\gamma}\rangle$ $\lvert g\rangle\Leftrightarrow\lvert\tilde{g}\rangle$	No boson singlets/ doublets $W^+\Leftrightarrow\tilde{W}^+$ $Z\Leftrightarrow\tilde{Z}$ $\gamma\Leftrightarrow\tilde{\gamma}$ $g\Leftrightarrow\tilde{g}$
Antiparticles (RH); anti-sparticles (RH)	$\lvert W^-\rangle$ & $\lvert\tilde{W}^-\rangle$ $\lvert\bar{g}\rangle$ & $\lvert\bar{\tilde{g}}\rangle$ Z,γ = antipartics	$\begin{pmatrix}W^-\\ \tilde{W}^-\end{pmatrix},\begin{pmatrix}\bar{\tilde{g}}\\ \bar{g}\end{pmatrix}$	$Q_2'^{\dagger}\lvert W^-\rangle=A'\lvert\tilde{W}^-\rangle$ $Q_2'\lvert\tilde{W}^-\rangle=B'\lvert W^-\rangle$ $Q_2'^{\dagger}\lvert\bar{g}\rangle=A'\lvert\bar{\tilde{g}}\rangle$ $Q_2'\lvert\bar{\tilde{g}}\rangle=B'\lvert\bar{g}\rangle$	$\lvert W^-\rangle\Leftrightarrow\lvert\tilde{W}^-\rangle$ $\lvert\bar{g}\rangle\Leftrightarrow\lvert\bar{\tilde{g}}\rangle$	$W^-\Leftrightarrow\tilde{W}^-$ $\bar{g}\Leftrightarrow\bar{\tilde{g}}$

Right Helicity Vector Bosons

R helicity massless gauge bosons (for the *W*s and gluons) follow in parallel, as RC followed LC for matter supermultiplets.

Wholeness Chart 6-7. N= 1 Gravity (j = 2) Superdoublets

Left Helicity Tensor (Gravity) Bosons

	Superspace		SUSY Operator	SU(2) Space	
	States	Fields		States	Fields
Particle (LH); sparticle (LH)	$\lvert G\rangle$ & $\lvert\tilde{G}\rangle$ Spin 2 graviton & spin 3/2 gravitino	Doublets $\begin{pmatrix}G\\ \tilde{G}\end{pmatrix}$	$Q_2''\lvert G\rangle=A''\lvert\tilde{G}\rangle$ $Q_2''^{\dagger}\lvert\tilde{G}\rangle=B''\lvert G\rangle$	$\lvert G\rangle\Leftrightarrow\lvert\tilde{G}\rangle$	$G\Leftrightarrow\tilde{G}$
Antipart (RH)	Graviton is its own anti-particle.				

Right Helicity Tensor (Gravity) Bosons

Opposite helicity parallels RC compared to LC in matter supermultiplets

Wholeness Chart 6-8. N= 2 Supermultiplets

Left Chiral Superquartets

	Superspace		SUSY Operator	SU(2) Space	
	States	Fields		States	Fields
Particles (LH), sparticles. particles (RH)	$\left\lvert+\tfrac{1}{2}\right\rangle_R,\lvert 0\rangle,$ $\left\lvert-\tfrac{1}{2}\right\rangle_L,\lvert 0^{\dagger}\rangle$ or $\lvert\psi\rangle,\lvert\phi\rangle,$ $\lvert\chi\rangle,\lvert\phi^{\dagger}\rangle$	Quartet $\begin{pmatrix}\psi\\ \phi\\ \chi\\ \phi^{\dagger}\end{pmatrix}$	$Q_1\left\lvert+\tfrac{1}{2}\right\rangle_R=A\lvert 0\rangle$ $Q_2\lvert 0\rangle=B\left\lvert-\tfrac{1}{2}\right\rangle_L$ $Q_3\left\lvert-\tfrac{1}{2}\right\rangle_L=C\lvert 0^{\dagger}\rangle$ $Q_4\lvert 0^{\dagger}\rangle=D\left\lvert+\tfrac{1}{2}\right\rangle_R$ Reverse operations not shown	$\left\lvert+\tfrac{1}{2}\right\rangle_R\Leftrightarrow\lvert 0\rangle$ $\lvert 0\rangle\Leftrightarrow\left\lvert-\tfrac{1}{2}\right\rangle_L$ $\left\lvert-\tfrac{1}{2}\right\rangle_L\Leftrightarrow\lvert 0^{\dagger}\rangle$ $\lvert 0^{\dagger}\rangle\Leftrightarrow\left\lvert+\tfrac{1}{2}\right\rangle_R$	LC Doublets $\begin{pmatrix}\chi_u\\ \chi_d\end{pmatrix}\Leftrightarrow\begin{pmatrix}\phi_u^{\dagger}\\ \phi_d^{\dagger}\end{pmatrix}$ RC Singlets $\psi_u\Leftrightarrow\phi_u$ $\psi_d\Leftrightarrow\phi_d$
Note	Under SUSY transformations, all SM couplings need to stay the same. In other words, in the SUSY quartet, each component (each field type) has to interact in $SU(3)$, $SU(2)$, and $U(1)$ interactions in the same way. But in weak interactions (SU(2) space), the LC χ and the RC ψ interact in different ways. The RC ψ have zero coupling to the *W*s, for example, whereas the LC χ do not. Similar effects occur for $N > 2$ SUSY, where multiplets have more field components than N=1 (doublets with two components) and N=2 (quartets with four components).				
Conclusion:	Only $N = 1$ SUSY is consistent with the standard model (at least without considerable manipulation of the theory).				

6.6.1 N > 2 SUSY

Similar effects as in $N = 2$ SUSY theory result from theories of N greater than 2, so, at least in the simplest form, they cannot be used as an adjunct to the standard model.

For completeness, we note that (6-10) can be generalized for any N to

$$\begin{aligned} &\left[Q_a^A, Q_b^{B\dagger}\right] = \delta^{AB}\left(\sigma^\mu\right)_{ab} P_\mu && A,B = 1,\ldots,N \\ &\left[Q_a^A, Q_b^{B\dagger}\right]_+ = \varepsilon_{ab} Z^{AB} \qquad \varepsilon_{11} = \varepsilon_{22} = 0 \quad \varepsilon_{12} = -1 \quad \varepsilon_{21} = +1 && Z^{AB} \text{ anti-sym in } A,B \end{aligned} \tag{6-11}$$

Note, in passing, that in $N = 8$ SUSY operators take an RH graviton and step it down sequentially to helicities of +3/2, +1, +½, 0, −½, −1, −3/2, and −2. $N > 8$ theories would need to act on spins greater than 2. Since $N = 8$ SUSY includes all possible spins in the universe as we know it, $N > 8$ theories are even more problematic than $2 \le N \le 8$ theories.

6.7 SUSY Currents and Charges Derivation

6.7.1 Left Chiral Fermions and Right Chiral Antifermions

We follow steps 1 to 7 in Sect. 6.1, the steps to develop any type of QFT (e.g., strong, weak, e/m, SUSY).

Preliminary Note A:

Recall that our derivation of the Euler-Lagrange equation of motion started from the postulate that, for fixed initial and final field configurations, the action is stationary, i.e., a variation in the Lagrangian (density) $\mathcal{L}$ left the action S invariant.

$$S = \int \mathcal{L} d^4x \qquad \delta S = \delta \int \mathcal{L} d^4x = \int \delta\mathcal{L} d^4x = 0 \qquad \text{true for } \delta\mathcal{L} = 0, \text{ i.e., } \mathcal{L} = \mathcal{L}_{sym}. \tag{6-12}$$

So, under a given transformation, represented by δ, the action is unchanged if $\mathcal{L}$ is symmetric (invariant) under that same transformation, i.e., $\delta\mathcal{L} = 0$. From this relation, we deduced the Euler-Lagrange equation.

However, note that what happens to a total derivative term inside the LH relation integral of (6-12), which is similar to the 1D case,

$$\int_{x=a}^{x=b} \frac{\partial}{\partial x} f(x) d^4x = \int_{f(a)}^{f(b)} df = \left(f(b) - f(a)\right) = \text{ constant} \tag{6-13}$$

In 4D, we have

$$\int_a^b \partial_\mu f^\mu\left(x^\nu\right) d^4x = \left(f^\mu(b) - f^\mu(a)\right) = \text{ constant for fixed initial and final conditions}. \tag{6-14}$$

So, any variation of the constant (6-14) is zero.

$$\delta \int_a^b \partial_\mu f^\mu\left(x^\nu\right) d^4x = \delta\left(f^\mu(b) - f^\mu(a)\right) = \delta(\text{constant}) = 0 \tag{6-15}$$

Note what happens if the variation in the action, instead of (6-12), looks like the following.

$$\delta S = \delta \int \mathcal{L} d^4x = \delta \int \left(\mathcal{L}_{sym} + \left(\partial_\mu f^\mu\right)\right) d^4x = \underbrace{\overbrace{\int \delta\mathcal{L}_{sym} d^4x}^{\int \frac{\partial \mathcal{L}_{sym}}{\partial \alpha} \alpha d^4x}}_{=0} + \overbrace{\int \delta\left(\partial_\mu f^\mu\right) d^4x}^{\int \frac{\partial\left(\partial_\mu f^\mu\right)}{\partial \alpha} \alpha d^4x} = \alpha \frac{\partial}{\partial \alpha} \underbrace{\int \partial_\mu f^\mu\, d^4x}_{\text{constant}} = 0. \tag{6-16}$$

In fact, anytime we have a total derivative like $\partial_\mu f^\mu$ in the variation of the Lagrangian, it will leave the action unchanged. For example, where C is any constant,

$$\delta\mathcal{L} = \delta\mathcal{L}_{sym} + C\partial_\mu f^\mu \qquad \rightarrow \qquad \delta S = \delta \int \mathcal{L} d^4x = \int \left(\delta\mathcal{L}_{sym} + C\left(\partial_\mu f^\mu\right)\right) d^4x = 0 \tag{6-17}$$

<u>Bottom line</u>: We could have a Lagrangian that is not symmetric, where the variation of the non-symmetric part comprises a total derivative times a constant, and still have an invariant action. And thus, the theory is unchanged.

A symmetric Lagrangian is sufficient for an invariant action, but not necessary.

Preliminary Note B:

Recall, for reference, the Euler-Lagrange equation

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\right)-\frac{\partial \mathcal{L}}{\partial \phi^{r}}=0\,. \qquad (6\text{-}18)$$

We now repeat the derivation of the conserved current for (6-17) instead of (6-12). See Klauber Vol. 1, Sect. 6.5.3, pgs 173-174, where the following is done for $f^{\mu}=0$.

$$\mathcal{L}=\mathcal{L}\left(\phi^{r},\phi^{r}{}_{,\mu}\right) \text{ symmetric in } \alpha, \text{ then}$$

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\alpha}\delta\alpha=0=\underbrace{\frac{\partial\mathcal{L}}{\partial\phi^{r}}}_{\substack{\text{use Euler-}\\ \text{Lagrange}\\ \text{equation}}}\frac{\partial\phi^{r}}{\partial\alpha}\delta\alpha+\frac{\partial\mathcal{L}}{\partial\phi^{r}{}_{,\mu}}\frac{\partial\phi^{r}{}_{,\mu}}{\partial\alpha}\delta\alpha \qquad (6\text{-}19)$$

$$\delta\mathcal{L}=0=\left(\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial\mathcal{L}}{\partial\phi^{r}{}_{,\mu}}\right)\frac{\partial\phi^{r}}{\partial\alpha}+\frac{\partial\mathcal{L}}{\partial\phi^{r}{}_{,\mu}}\underbrace{\frac{\partial}{\partial x^{\mu}}\frac{\partial\phi^{r}}{\partial\alpha}}_{\frac{\partial\phi^{r}{}_{,\mu}}{\partial\alpha}}\right)\delta\alpha=\frac{\partial}{\partial x^{\mu}}\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi^{r}{}_{,\mu}}\frac{\partial\phi^{r}}{\partial\alpha}\right)}_{j^{\mu}}\delta\alpha=0 \qquad (6\text{-}20)$$

For arbitrary variation α, we must have, for our 4-current j^{μ},

$$\partial_{\mu}j^{\mu}=0 \quad\rightarrow\quad \int_{\substack{\text{all}\\ \text{space}}} j^{0}d^{3}x=Q'=\text{constant in time}\,. \qquad (6\text{-}21)$$

But note that we can use (6-17) in (6-12) and leave the theory intact. We take the constant C = the transformation parameter α here (which is constant over spacetime, though varied in the transformation), because doing so will turn out to be valuable.

$$0=\delta S=\int\delta\mathcal{L}d^{4}x=\int\left(\delta\mathcal{L}_{sym}+\partial_{\mu}f^{\mu}\alpha\right)d^{4}x=\int\left(\left(\partial_{\mu}j^{\mu}\right)\alpha+\left(\partial_{\mu}f^{\mu}\right)\alpha\right)d^{4}x$$
$$=\int\partial_{\mu}\underbrace{\left(j^{\mu}+f^{\mu}\right)}_{j^{\mu}_{altern}}\alpha d^{4}x \quad\rightarrow\quad \partial_{\mu}j^{\mu}_{altern}=0. \qquad , \qquad (6\text{-}22)$$

Bottom line: Thus, we can have a valid, alternative current, where

$$j^{\mu}_{altern}=j^{\mu}\pm f^{\mu} \qquad \text{where } \delta\mathcal{L}=\frac{\partial\mathcal{L}_{sym}}{\partial\alpha}\alpha\pm\partial_{\mu}f^{\mu}\alpha\,. \qquad (6\text{-}23)$$

Step #1: Guess at a suitable SUSY Lagrangian

Where ϕ is a scalar field, χ is a LC spinor field, and F is an auxiliary field that, as we will see, is needed to have an invariant action,

$$\mathcal{L}=\partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi+\chi^{\dagger}i\bar{\sigma}^{\mu}\partial_{\mu}\chi+F^{\dagger}F=\phi^{\dagger}_{,\mu}\phi^{,\mu}+\chi^{\dagger}i\bar{\sigma}^{\mu}\chi_{,\mu}+F^{\dagger}F \qquad \text{Aitchison (4.107) [66].(6-24)}$$

Note that there is no kinetic term for F, so it doesn't change in time and is innocuous to the theory. In fact, in the Euler-Lagrange equation for F, we find the equation of motion for F as

$$F=0 \quad \text{(equation of motion for } F\text{).} \qquad (6\text{-}25)$$

It doesn't show up in the real world, but helps us behind the scenes to make the action invariant. Note also that the scalar ϕ will be the superpartner of the LC spinor χ.

Step #2. Find an Internal Transformation on SUSY Fields for Symmetry of S ($\delta S=0$)

The Transformations on Fields

As we will see, the following transformation set gives us an invariant SUSY action.

$$\delta_\xi \phi = \xi \cdot \chi = \xi^T \left(-i\sigma_2\right)\chi \qquad \delta_\xi \phi^\dagger = \bar{\chi} \cdot \bar{\xi} = \chi^\dagger i\sigma_2 \xi^* \qquad \text{Aitchison (4.119) \{68\}} \quad (6\text{-}26)$$

$$\delta_\xi \chi = -i\sigma^\mu \left(i\sigma_2\right)\xi^* \partial_\mu \phi + \xi F \qquad \delta_\xi \chi^\dagger = i\partial_\mu \phi^\dagger \xi^T \left(-i\sigma_2\right)\sigma^\mu + F^\dagger \xi^\dagger \qquad \text{Aitchison (4.121) \{68\}} \quad (6\text{-}27)$$

$$\delta_\xi F = -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \qquad \delta_\xi F^\dagger = i\partial_\mu \chi^\dagger \bar{\sigma}^\mu \xi \qquad \text{Aitchison (4.120) \{68\}} \quad (6\text{-}28)$$

Note that (6-26) and (6-27) can be expressed via a matrix in (2D) superspace,

$$\delta_\xi \begin{bmatrix} \phi \\ \chi \end{bmatrix} = \begin{bmatrix} & \xi\bullet \\ -i\sigma^\mu \bar{\xi}\partial_\mu & \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} + \begin{bmatrix} 0 \\ \xi F \end{bmatrix}, \tag{6-29}$$

where, if we are talking about particles one might see in the real world, we can ignore the last column vector and $\begin{bmatrix} \phi \\ \chi \end{bmatrix}$ is a doublet in superspace.

<u>Proving the Action is Invariant</u>

From (6-24), we have

$$\begin{aligned} \delta_\xi \mathcal{L} &= \delta_\xi \left(\phi^\dagger_{,\mu}\phi^{,\mu}\right) + \delta_\xi \left(\chi^\dagger i\bar{\sigma}^\mu \chi_{,\mu}\right) + \delta_\xi \left(F^\dagger F\right) \\ &= \left(\delta_\xi \phi^\dagger_{,\mu}\right)\phi^{,\mu} + \phi^\dagger_{,\mu}\left(\delta_\xi \phi^{,\mu}\right) + \left(\delta_\xi \chi^\dagger\right) i\bar{\sigma}^\mu \chi_{,\mu} + \chi^\dagger i\bar{\sigma}^\mu \left(\delta_\xi \chi_{,\mu}\right) + \left(\delta_\xi F^\dagger\right)F + F^\dagger\left(\delta_\xi F\right) \\ &= \left(\partial_\mu\left(\delta_\xi \phi^\dagger\right)\right)\partial^\mu \phi + \left(\partial_\mu \phi^\dagger\right)\partial^\mu\left(\delta_\xi \phi\right) + \left(\delta_\xi \chi^\dagger\right) i\bar{\sigma}^\mu \chi_{,\mu} + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu\left(\delta_\xi \chi\right) + \left(\delta_\xi F^\dagger\right)F + F^\dagger\left(\delta_\xi F\right). \end{aligned} \tag{6-30}$$

With the transformations of (6-26) to (6-28), (6-30) becomes

$$\begin{aligned} \delta_\xi \mathcal{L} = &\left(\partial_\mu\left(\chi^\dagger i\sigma_2 \xi^*\right)\right)\partial^\mu \phi + \left(\partial_\mu \phi^\dagger\right)\partial^\mu\left(\xi^T\left(-i\sigma_2\right)\chi\right) + \left(i\partial_\mu \phi^\dagger \xi^T\left(-i\sigma_2\right)\sigma^\mu + F^\dagger \xi^\dagger\right) i\bar{\sigma}^\nu \partial_\nu \chi \\ &+ \chi^\dagger i\bar{\sigma}^\nu \partial_\nu\left(-i\sigma^\mu\left(i\sigma_2\right)\xi^* \partial_\mu \phi + \xi F\right) + \left(i\partial_\mu \chi^\dagger \bar{\sigma}^\mu \xi\right)F + F^\dagger\left(-i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi\right). \end{aligned} \tag{6-31}$$

Separating out the terms with F fields, we have

$$\begin{aligned} \delta_\xi \mathcal{L} = &\overbrace{\left(\partial_\mu\left(\chi^\dagger i\sigma_2 \xi^*\right)\right)\partial^\mu \phi}^{\boxed{A}} + \overbrace{\left(\partial_\mu \phi^\dagger\right)\partial^\mu\left(\xi^T\left(-i\sigma_2\right)\chi\right)}^{\boxed{B}} \\ &+ \overbrace{\left(i\partial_\mu \phi^\dagger \xi^T\left(-i\sigma_2\right)\sigma^\mu\right) i\bar{\sigma}^\nu \partial_\nu \chi}^{\boxed{C}} + \overbrace{\chi^\dagger i\bar{\sigma}^\nu \partial_\nu\left(-i\sigma^\mu\left(i\sigma_2\right)\xi^* \partial_\mu \phi\right)}^{\boxed{D}} \\ &+ \overbrace{\left(F^\dagger \xi^\dagger\right) i\bar{\sigma}^\mu \partial_\mu \chi}^{\text{cancels with last term}} + \overbrace{\chi^\dagger i\bar{\sigma}^\mu \partial_\mu\left(\xi F\right) + \left(i\partial_\mu \chi^\dagger \bar{\sigma}^\mu \xi\right)F}^{\boxed{E} = \left(\partial_\mu\left(\chi^\dagger i\bar{\sigma}^\mu F\right)\right)\xi} + \overbrace{F^\dagger\left(-i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi\right)}^{\text{cancels}}. \end{aligned} \tag{6-32}$$

The variation (6-32) is not zero, meaning the Lagrangian is not symmetric under this transformation set. However, all is not lost, as we are about to see.

$\boxed{E}$ is a total derivative times the parameter being varied ξ, which, as we saw above in (6-17), leaves the action invariant. The Lagrangian may not be symmetric, in part due to this term that does not vanish in (6-32), but the overriding principle of action invariance is still upheld by this term. Note this makes the field F a non-factor physically, since its variation does not affect the action, and since, as we saw above, its equation of motion is $F = 0$.

Let's now look at the terms in ξ^* in (6-32), i.e., terms $\boxed{A}$ and $\boxed{D}$.

$$\begin{aligned} \boxed{A} + \boxed{D} &= \left(\partial_\mu\left(\chi^\dagger i\sigma_2 \xi^*\right)\right)\partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\nu \partial_\nu\left(-i\sigma^\mu\left(i\sigma_2\right)\xi^* \partial_\mu \phi\right) \\ &= \left(\partial_\mu\left(\chi^\dagger i\sigma_2 \xi^*\right)\right)\partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\nu \partial_\nu\left(\sigma^\mu \partial_\mu \phi\left(\sigma_2\right)\xi^*\right). \end{aligned} \tag{6-33}$$

Note that, in the last term in (6-33),

$$
\begin{aligned}
\bar{\sigma}^{\nu}\partial_{\nu}\sigma^{\mu}\partial_{\mu} &= \left(I\partial_0 - \sigma^1\partial_1 - \sigma^2\partial_2 - \sigma^3\partial_3\right)\left(I\partial_0 + \sigma^1\partial_1 + \sigma^2\partial_2 + \sigma^3\partial_3\right) \\
&= \partial_0^2 + \sigma^1\partial_0\partial_1 + \sigma^2\partial_0\partial_2 + \sigma^3\partial_0\partial_3 \\
&\quad -\sigma^1\partial_0\partial_1 - \sigma^1\sigma^1\partial_1\partial_1 - \sigma^1\sigma^2\partial_1\partial_2 - \sigma^1\sigma^3\partial_1\partial_3 \\
&\quad -\sigma^2\partial_0\partial_2 - \sigma^2\sigma^1\partial_1\partial_2 - \sigma^2\sigma^2\partial_2\partial_2 - \sigma^2\sigma^3\partial_2\partial_3 \\
&\quad -\sigma^3\partial_0\partial_3 - \sigma^3\sigma^1\partial_1\partial_3 - \sigma^3\sigma^2\partial_3\partial_2 - \sigma^3\sigma^3\partial_3\partial_3
\end{aligned}
\tag{6-34}
$$

$$
\begin{aligned}
\bar{\sigma}^{\nu}\partial_{\nu}\sigma^{\mu}\partial_{\mu} &= \partial_0^2 - \sigma^1\sigma^1\partial_1\partial_1 - \sigma^1\sigma^2\partial_1\partial_2 - \sigma^1\sigma^3\partial_1\partial_3 - \sigma^2\sigma^1\partial_1\partial_2 - \sigma^2\sigma^2\partial_2\partial_2 - \sigma^2\sigma^3\partial_2\partial_3 \\
&\qquad - \sigma^3\sigma^1\partial_1\partial_3 - \sigma^3\sigma^2\partial_3\partial_2 - \sigma^3\sigma^3\partial_3\partial_3 \\
&= \partial_0^2 - \underbrace{\sigma^1\sigma^1}_{I}\partial_1\partial_1 - \underbrace{\sigma^2\sigma^2}_{I}\partial_2\partial_2 - \underbrace{\sigma^3\sigma^3}_{I}\partial_3\partial_3 - \underbrace{\left[\sigma^1,\sigma^2\right]_+}_{0}\partial_1\partial_2 - \underbrace{\left[\sigma^2,\sigma^3\right]_+}_{0}\partial_2\partial_3 - \underbrace{\left[\sigma^1,\sigma^3\right]_+}_{0}\partial_1\partial_3 \\
&= \partial_0^2 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3 = \partial_0^2 + \partial^i\partial_i = \partial^{\mu}\partial_{\mu}\,.
\end{aligned}
\tag{6-35}
$$

With the result of (6-35) in (6-33), we have

$$
\boxed{A} + \boxed{D} = \left(\partial_{\mu}\left(\chi^{\dagger} i\sigma_2\xi^*\right)\right)\partial^{\mu}\phi + i\chi^{\dagger}\partial^{\mu}\left(\partial_{\mu}\phi i\sigma_2\xi^*\right) = \partial_{\mu}\left(\left(\chi^{\dagger} i\sigma_2\right)\partial^{\mu}\phi\right)\xi^*\,. \tag{6-36}
$$

$\boxed{A}$ and $\boxed{D}$ in (6-32) comprise a total derivative times the variation parameter ξ^*, so it too leaves the action invariant, and thus, does not, in the final analysis, doom our approach.

Let's now look at the terms in ξ^{T} in (6-32), i.e., terms $\boxed{B}$ and $\boxed{C}$, where in the second line we use (6-35).

$$
\begin{aligned}
\boxed{B} + \boxed{C} &= \left(\partial_{\mu}\phi^{\dagger}\right)\partial^{\mu}\left(\xi^T\left(-i\sigma_2\right)\chi\right) + \left(\partial_{\mu}\phi^{\dagger}\xi^T\left(i\sigma_2\right)\sigma^{\mu}\right)\bar{\sigma}^{\nu}\partial_{\nu}\chi \\
&= \left(\partial_{\mu}\phi^{\dagger}\right)\partial^{\mu}\left(\xi^T\left(-i\sigma_2\right)\chi\right) + \partial_{\mu}\left(\phi^{\dagger}\xi^T\left(i\sigma_2\right)\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) - \left(\phi^{\dagger}\xi^T\left(i\sigma_2\right)\right)\underbrace{\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\mu}\partial_{\nu}}_{\partial_{\mu}\partial^{\mu}}\chi \\
&= \left(\partial_{\mu}\phi^{\dagger}\right)\partial^{\mu}\left(\xi^T\left(-i\sigma_2\right)\chi\right) + \xi^T\partial_{\mu}\left(\phi^{\dagger}\left(i\sigma_2\right)\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) - \left(\phi^{\dagger}\xi^T\left(i\sigma_2\right)\right)\partial_{\mu}\left(\partial^{\mu}\chi\right).
\end{aligned}
\tag{6-37}
$$

The second term in (6-37) is a total derivative. The first and last terms combine to form a total derivative times the variation parameter ξ^{T}.

$$
\begin{aligned}
\boxed{B} + \boxed{C} &= \left(\partial_{\mu}\phi^{\dagger}\right)\partial^{\mu}\left(\xi^T\left(-i\sigma_2\right)\chi\right) + \xi^T\partial_{\mu}\left(\phi^{\dagger}\left(i\sigma_2\right)\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) - \left(\phi^{\dagger}\xi^T\left(i\sigma_2\right)\right)\partial_{\mu}\left(\partial^{\mu}\chi\right) \\
&= -\xi^T\left(\partial_{\mu}\phi^{\dagger}\right)\left(\left(i\sigma_2\right)\partial^{\mu}\chi\right) - \xi^T\left(\phi^{\dagger}\left(i\sigma_2\right)\right)\partial_{\mu}\left(\partial^{\mu}\chi\right) + \xi^T\partial_{\mu}\left(\phi^{\dagger}\left(i\sigma_2\right)\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) \\
&= \xi^T\partial_{\mu}\left(\phi^{\dagger}\left(\left(-i\sigma_2\right)\partial^{\mu}\chi\right)\right) + \xi^T\partial_{\mu}\left(\phi^{\dagger}\left(i\sigma_2\right)\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) = \xi^T\partial_{\mu}\left(-\phi^{\dagger} i\sigma_2\partial^{\mu}\chi + \phi^{\dagger} i\sigma_2\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right).
\end{aligned}
\tag{6-38}
$$

Thus, we have, from (6-32), (6-36), and (6-38),

$$
\delta_{\xi}\mathcal{L} = \partial_{\mu}\left(\left(\chi^{\dagger} i\sigma_2\right)\partial^{\mu}\phi\right)\xi^* + \xi^T\partial_{\mu}\left(-\phi^{\dagger} i\sigma_2\partial^{\mu}\chi + \phi^{\dagger} i\sigma_2\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\nu}\chi\right) + \left(\partial_{\mu}\left(\chi^{\dagger} i\bar{\sigma}^{\mu}F\right)\right)\xi \tag{6-39}
$$

And thus, all surviving terms in (6-32) are total derivatives times a variation parameter, so the action is invariant.

<u>Bottom line</u>: The SUSY Lagrangian is not symmetric, but because its variation is a sum of total derivatives times a variation parameter, the SUSY action is symmetric (invariant). And so, we can have a consistent SUSY theory.

<u>Step #3. Commutation Relations for Transformation Matrices of Superdoublet Space</u>

This is not so relevant in supersymmetry, though it played a key role in $SU(n)$ theories of the standard model. There it entailed the Lie Algebra commutation relations for $SU(n)$ space operations on fields.

<u>Step #4. Noether's theorem to find the conserved 4-currents</u>

Recall the 4-current from Noether's theorem, where ξ is the small parameter varied in the symmetry transformation set and a primed field represents the transformed field,

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'}{\partial \xi} \qquad \xi \text{ real}. \tag{6-40}$$

In our case, we have to be a bit careful, as ξ is a spinor quantity, and so is a complex number. Thus, to be complete, we need to consider j^{μ} as the sum of a 4-current and its complex conjugate transpose. We'll designate this with subscript *altern*.

$$j^{\mu}_{altern} = j^{\mu} + j^{\mu\dagger} = \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'}{\partial \xi^{*}} \qquad \xi \text{ complex and } \phi^{r} = \phi, \phi^{\dagger}, \chi, \chi^{\dagger}, F, F^{\dagger} \text{ for the SUSY Lagrangian} \tag{6-41}$$

Also recall that (6-40) was derived from the stationary action principle, where $\delta S = 0$, and where we assumed we had no total derivative term, such as $\partial_{\mu} f^{\mu}$ in (6-23). If we have such a total derivative term, we find, where by convention one uses a minus sign in front of f^{μ} and multiplies what will be the SUSY current by the constant i,

$$j^{\mu}_{altern} = j^{\mu} + j^{\mu\dagger} = \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'}{\partial \xi^{*}} - f^{\mu} \qquad \text{with } \partial_{\mu} f^{\mu} \neq 0 \text{ in } \delta\mathcal{L}. \tag{6-42}$$

Thus,

$$\begin{aligned} j^{\mu}_{altern} = \frac{\partial \mathcal{L}}{\partial \phi^{r}_{,\mu}} \frac{\partial \phi'^{r}}{\partial \alpha} &= \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \frac{\partial \phi'}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}_{,\mu}} \frac{\partial \phi'^{\dagger}}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \chi_{,\mu}} \frac{\partial \chi'}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial \chi^{\dagger}_{,\mu}} \frac{\partial \chi'^{\dagger}}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial F_{,\mu}} \frac{\partial F'}{\partial \xi} + \frac{\partial \mathcal{L}}{\partial F^{\dagger}_{,\mu}} \frac{\partial F'}{\partial \xi} \\ &+ \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \frac{\partial \phi'}{\partial \xi^{*}} + \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}_{,\mu}} \frac{\partial \phi'^{\dagger}}{\partial \xi^{*}} + \frac{\partial \mathcal{L}}{\partial \chi_{,\mu}} \frac{\partial \chi'}{\partial \xi^{*}} + \frac{\partial \mathcal{L}}{\partial \chi^{\dagger}_{,\mu}} \frac{\partial \chi'^{\dagger}}{\partial \xi^{*}} + \frac{\partial \mathcal{L}}{\partial F_{,\mu}} \frac{\partial F'}{\partial \xi^{*}} + \frac{\partial \mathcal{L}}{\partial F^{\dagger}_{,\mu}} \frac{\partial F'}{\partial \xi^{*}} - f^{\mu}. \end{aligned} \tag{6-43}$$

Keep in mind, in the following, that the complex conjugate transpose of any Pauli matrix equals the original Pauli matrix, and also, that ϕ is a scalar, not a spinor (i.e., Pauli matrices don't operate on ϕ or its derivatives, so the order in which ϕ, or its derivative, appears in a string of spinor fields and operators is unimportant).

From (6-24) and (6-26) to (6-28), for the first row of (6-43), where we repeat (6-24) for convenience,

$$\begin{aligned} &\mathcal{L} = \phi^{\dagger,\mu}\phi_{,\mu} + \chi^{\dagger} i\bar{\sigma}^{\mu} \chi_{,\mu} + F^{\dagger}F \\ &\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \phi^{\dagger,\mu} \quad \frac{\partial \phi'}{\partial \xi} = \frac{\partial\left(\phi + \delta_{\xi}\phi\right)}{\partial \xi} = \frac{\partial\left(\phi + \xi^{T}\left(-i\sigma_{2}\right)\chi\right)}{\partial \xi} = \frac{\partial\left(\phi + \chi^{T}\left(i\sigma_{2}\right)\xi\right)}{\partial \xi} = \chi^{T}\left(i\sigma_{2}\right) \\ &\frac{\partial \mathcal{L}}{\partial \phi^{\dagger}_{,\mu}} = \phi^{,\mu} \quad \frac{\partial \phi'^{\dagger}}{\partial \xi} = \frac{\partial\left(\phi^{\dagger} + \delta_{\xi}\phi^{\dagger}\right)}{\partial \xi} = \frac{\partial\left(\phi + \chi^{\dagger}\left(i\sigma_{2}\right)\xi^{*}\right)}{\partial \xi} = 0 \\ &\frac{\partial \mathcal{L}}{\partial \chi_{,\mu}} = \chi^{\dagger} i\bar{\sigma}^{\mu} \quad \frac{\partial \chi'}{\partial \xi} = \frac{\partial\left(\chi + \delta_{\xi}\chi\right)}{\partial \xi} = \frac{\partial\left(\chi - i\sigma^{\nu}\left(i\sigma_{2}\right)\xi^{*}\phi_{,\nu} + \xi F\right)}{\partial \xi} = F \\ &\frac{\partial \mathcal{L}}{\partial \chi^{\dagger}_{,\mu}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial F_{,\mu}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial F^{\dagger}_{,\mu}} = 0 \end{aligned} \tag{6-44}$$

For the derivatives with respect to ξ^{*} of the second row of (6-43), the only ones not multiplied by zero are

$$\begin{aligned} &\frac{\partial \phi'}{\partial \xi^{*}} = \frac{\partial\left(\phi + \delta_{\xi}\phi\right)}{\partial \xi^{*}} = \frac{\partial\left(\phi + \xi^{T}\left(-i\sigma_{2}\right)\chi\right)}{\partial \xi^{*}} = \frac{\partial\left(\phi + \chi^{T}\left(i\sigma_{2}\right)\xi\right)}{\partial \xi^{*}} = 0 \\ &\frac{\partial \phi'^{\dagger}}{\partial \xi^{*}} = \frac{\partial\left(\phi^{\dagger} + \delta_{\xi}\phi^{\dagger}\right)}{\partial \xi^{*}} = \frac{\partial\left(\phi + \chi^{\dagger}\left(i\sigma_{2}\right)\xi^{*}\right)}{\partial \xi^{*}} = \chi^{\dagger}\left(i\sigma_{2}\right) \qquad . \\ &\frac{\partial \chi'}{\partial \xi^{*}} = \frac{\partial\left(\chi + \delta_{\xi}\chi\right)}{\partial \xi^{*}} = \frac{\partial\left(\chi - i\sigma^{\nu}\left(i\sigma_{2}\right)\xi^{*}\phi_{,\nu} + \xi F\right)}{\partial \xi^{*}} = -i\sigma^{\nu}\left(i\sigma_{2}\right)\phi_{,\nu} \end{aligned} \tag{6-45}$$

With (6-44) and (6-45) into (6-43), we have

$$j^{\mu}_{altern} = \chi^T\left(i\sigma_2\right)\phi^{,\mu\dagger} + \chi^\dagger\left(i\sigma_2\right)\phi^{,\mu} + \chi^\dagger i\bar{\sigma}^\mu F + \chi^\dagger i\bar{\sigma}^\mu\left(-i\sigma^\nu\right)\left(i\sigma_2\right)\phi_{,\nu} - f^\mu . \tag{6-46}$$

Now, take f^μ as the quantities being operated on by ∂_μ in (6-39)

$$f^\mu = \left(\chi^\dagger i\sigma_2\right)\phi^{,\mu} + \left(i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu}\right) + \underbrace{\left(-i\sigma_2\right)\chi}_{\chi^T\left(i\sigma_2\right)}\phi^{,\mu\dagger} + \chi^\dagger i\bar{\sigma}^\mu F \tag{6-47}$$

This makes (6-46) into

$$\begin{aligned} j^{\mu}_{altern} &= \chi^T\left(i\sigma_2\right)\phi^{,\mu\dagger} + \chi^\dagger\left(i\sigma_2\right)\phi^{,\mu} + \chi^\dagger i\bar{\sigma}^\mu F + \chi^\dagger i\bar{\sigma}^\mu\left(-i\sigma^\nu\right)\left(i\sigma_2\right)\phi_{,\nu} - f^\mu \\ &\qquad -\left(\chi^\dagger i\sigma_2\right)\phi^{,\mu} - \left(i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu}\right) - \chi^T\left(i\sigma_2\right)\phi^{,\mu\dagger} - \chi^\dagger i\bar{\sigma}^\mu F \\ &= \chi^\dagger i\bar{\sigma}^\mu\left(-i\sigma^\nu\right)\left(i\sigma_2\right)\phi_{,\nu} - \left(i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu}\right) = i\chi^\dagger\bar{\sigma}^\mu\sigma^\nu\sigma_2\phi_{,\nu} - i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu} \\ &= \left(i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu}\right)^\dagger + i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu} . \end{aligned} \tag{6-48}$$

Comparing (6-48) to the LHS of (6-42), we find

$$j^{\mu}_{altern} = j^\mu + j^{\mu\dagger} = i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu} + \left(i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu}\right)^\dagger \quad\rightarrow\quad j^\mu = i\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu} . \tag{6-49}$$

In defining the SUSY current, we drop the constant factor $i\sigma_2$ (which is irrelevant for the defining relation $\partial_\mu j^\mu = 0$) and use an upper-case J, as Aitchison does. So,

$$j^\mu_{SUSY} = J^\mu = \sigma^\nu\bar{\sigma}^\mu\chi\phi^\dagger_{,\nu} \qquad \text{Aitchison (4.70) [60].} \tag{6-50}$$

In the above, we have sort of hidden it under the rug that χ actually has two (spinor) components and σ^ν and $\bar{\sigma}^\mu$ are 2X2 matrices with (counting zero value components) four components. Re-writing (6-50) with these component indices written out, we have

$$J^\mu_a = \sigma^\nu_{ab}\bar{\sigma}^\mu_{bc}\chi_c\phi^\dagger_{,\nu} , \tag{6-51}$$

so, there are really two *scalar* SUSY 4-currents,

$$J^\mu_1 = \sigma^\nu_{1b}\bar{\sigma}^\mu_{bc}\chi_c\phi^\dagger_{,\nu} \qquad \text{and} \qquad J^\mu_2 = \sigma^\nu_{2b}\bar{\sigma}^\mu_{bc}\chi_c\phi^\dagger_{,\nu} \tag{6-52}$$

where

$$\partial_\mu J^\mu_1 = 0 \qquad \text{and} \qquad \partial_\mu J^\mu_2 = 0 . \tag{6-53}$$

Equivalently,

$$\partial_\mu J^{\dagger\mu}_1 = 0 \qquad \text{and} \qquad \partial_\mu J^{\dagger\mu}_2 = 0 . \tag{6-54}$$

Step #5. Find the conserved charges Q_a by integrating each $J_a{}^0$ over all space

Our charge operators, representing conserved charge via (6-53) and (6-54), are defined as

$$Q_1 = \int J^0_1 d^3x = \int \sigma^\nu_{1b}\bar{\sigma}^0_{bc}\chi_c(x)\phi^\dagger_{,\nu}(x)d^3x = \int \sigma^\nu_{1b}\chi_b(x)\phi^\dagger_{,\nu}(x)d^3x \tag{6-55}$$

$$Q_2 = \int J^0_2 d^3x = \int \sigma^\nu_{2b}\bar{\sigma}^0_{bc}\chi_c(x)\phi^\dagger_{,\nu}(x)d^3x = \int \sigma^\nu_{2b}\chi_b(x)\phi^\dagger_{,\nu}(x)d^3x . \tag{6-56}$$

Note in Q_2, that $\phi^\dagger$ creates a scalar, and χ destroys a LC fermion. Compare with Wholeness Chart 6-1, where Q_2 turns a LC fermion into a scalar, so, intuitively, (6-56) makes some sense. As we develop Q_2 below, we will see this does indeed work out. We save similar comment on Q_1 until later.

Step #6. Determine the commutation relations for the charges Q_a

To determine the commutation and anti-commutation relations for the Q_a, we'll need to refer to the commutation/anticommutation relations for fields in QFT.

$$\left[\chi_a(t,\mathbf{x}),\chi_b^\dagger(t,\mathbf{y})\right]_+ = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}) \tag{6-57}$$

$$\left[\phi(t,\mathbf{x}),\phi^\dagger(t,\mathbf{y})\right] = i\delta(\mathbf{x}-\mathbf{y}) \tag{6-58}$$

All other scalar commutators equal zero. All other fermion anti-commutators equal zero.

$$\boxed{\left[Q_a,Q_b\right]_+ = 0}$$

To keep things simple to start, we'll take $a = 1$ and $b = 2$, then generalize.

$$\begin{aligned}\left[Q_1,Q_2\right]_+ = Q_1Q_2 + Q_2Q_1 = &\left(\int \sigma^\mu_{1c}\chi_c(x)\phi^\dagger_{,\mu}(x)d^3x\right)\left(\int \sigma^\nu_{2d}\chi_d(y)\phi^\dagger_{,\nu}(y)d^3y\right)\\ &+\left(\int \sigma^\nu_{2d}\chi_d(y)\phi^\dagger_{,\nu}(y)d^3y\right)\left(\int \sigma^\mu_{1c}\chi_c(x)\phi^\dagger_{,\mu}(x)d^3x\right)\end{aligned} \tag{6-59}$$

Note that the operation of a Pauli matrix on a spinor leaves us with a (different) spinor, which we will designate with a prime, and which will depend on the spacetime index on σ^μ.

$$\chi_1'^\mu = \sigma^\mu_{1c}\chi_c(x) \qquad\qquad \chi_2'^\nu = \sigma^\mu_{2c}\chi_c(x) \tag{6-60}$$

With that notation, the commutation property of the scalars, the anti-commutation property of fermions, and recognizing that the Greek index on a fermion field refers to the particular Pauli matrix used not a spacetime coordinate, (6-59) becomes

$$\begin{aligned}Q_1Q_2 + Q_2Q_1 &= \iint \left(\chi_1'^\mu\chi_2'^\nu\phi^\dagger_{,\mu}(x)\phi^\dagger_{,\nu}(y) + \chi_2'^\nu\chi_1'^\mu\phi^\dagger_{,\nu}(y)\phi^\dagger_{,\mu}(x)\right)d^3xd^3y\\ &= \iint \left(\chi_1'^\mu\chi_2'^\nu + \chi_2'^\nu\chi_1'^\mu\right)\phi^\dagger_{,\mu}(x)\phi^\dagger_{,\nu}(y)d^3xd^3y = \iint \underbrace{\left[\chi_1'^\mu,\chi_2'^\nu\right]_+}_{=0}\phi^\dagger_{,\mu}(x)\phi^\dagger_{,\nu}(y)d^3xd^3y.\end{aligned} \tag{6-61}$$

Thus,

$$\left[Q_1,Q_2\right]_+ = 0\,. \tag{6-62}$$

We can see that for any a and b combination, the anti-commutator in (6-61) would still be zero. So, in general,

$$\left[Q_a,Q_b\right]_+ = 0\,. \tag{6-63}$$

$$\boxed{\left[Q_a,Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab}P_\mu}$$

Again, we consider $a = 1$ and $b = 2$.

$$\begin{aligned}\left[Q_1,Q_2^\dagger\right]_+ = Q_1Q_2^\dagger + Q_2^\dagger Q_1 &= \left(\int \sigma^\mu_{1c}\chi_c(x)\phi^\dagger_{,\mu}(x)d^3x\right)\left(\int \chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger\phi_{,\nu}(y)d^3y\right)\\ &\quad+\left(\int \chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger\phi_{,\nu}(y)d^3y\right)\left(\int \sigma^\mu_{1c}\chi_c(x)\phi^\dagger_{,\mu}(x)d^3x\right)\\ &= \iint \left(\sigma^\mu_{1c}\chi_c(x)\chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger\phi^\dagger_{,\mu}(x)\phi_{,\nu}(y) + \chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger\sigma^\mu_{1c}\chi_c(x)\phi_{,\nu}(y)\phi^\dagger_{,\mu}(x)\right)d^3xd^3y\\ &= \iint \left(\sigma^\mu_{1c}\chi_c(x)\chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger + \chi_d^\dagger(y)\left(\sigma^\nu_{2d}\right)^\dagger\sigma^\mu_{1c}\chi_c(x)\right)\phi^\dagger_{,\mu}(x)\phi_{,\nu}(y)d^3xd^3y.\end{aligned} \tag{6-64}$$

Consider the case of (6-64) where $\mu = \nu = 0$, so $\sigma^0 = I$ (in spinor space).

$$\begin{aligned}\text{only } \mu=\nu=0 \text{ part of } \left[Q_1,Q_2^\dagger\right]_+ &= \iint \left(\sigma^0_{1c}\chi_c(x)\chi_d^\dagger(y)\left(\sigma^0_{2d}\right)^\dagger + \chi_d^\dagger(y)\left(\sigma^0_{2d}\right)^\dagger\sigma^0_{1c}\chi_c(x)\right)\phi^\dagger_{,0}(x)\phi_{,0}(y)d^3xd^3y\\ &= \iint \left(I_{1c}\chi_c(x)\chi_d^\dagger(y)I_{d2} + \chi_d^\dagger(y)I_{d2}I_{1c}\chi_c(x)\right)\phi^\dagger_{,0}(x)\phi_{,0}(y)d^3xd^3y\\ &= \iint \left(\chi_1(x)\chi_2^\dagger(y) + \chi_2^\dagger(y)\chi_1(x)\right)\phi^\dagger_{,0}(x)\phi_{,0}(y)d^3xd^3y\\ &= \iint \underbrace{\left[\chi_1(x)\chi_2^\dagger(y)\right]_+}_{\delta_{12}\delta(\mathbf{x}-\mathbf{y})=0}\phi^\dagger_{,0}(x)\phi_{,0}(y)d^3xd^3y\end{aligned} \tag{6-65}$$

For $\delta_{12} = \sigma^0_{12} = I_{12} = 0$, this can be written as

$$\text{only } \mu = \nu = 0 \text{ part of } \left[Q_1, Q_2^\dagger\right]_+ = \iint \sigma^0_{12}\phi^\dagger_{,0}(x)\phi_{,0}(y)\,d^3x d^3y\,. \tag{6-66}$$

Note that

$$\begin{aligned}\phi_{,0}(y) &= \sum_{\mathbf{k}} \frac{\partial}{\partial x^0}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}e^{-iky} + b^\dagger_{\mathbf{k}}e^{iky}\right) = \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}e^{-iky} - b^\dagger_{\mathbf{k}}e^{iky}\right) \\ \phi^\dagger_{,0}(y) &= \sum_{\mathbf{k}'} \frac{\partial}{\partial x^0}\frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a^\dagger_{\mathbf{k}'}e^{iky} + b_{\mathbf{k}'}e^{-iky}\right) = \sum_{\mathbf{k}'} \frac{i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a^\dagger_{\mathbf{k}'}e^{iky} - b_{\mathbf{k}'}e^{-iky}\right)\end{aligned} \tag{6-67}$$

The full evaluation of the following relation (6-68) parallels that of Klauber, Vol. 1, Sect. 3.4.1, pg. 53.

$$\int \phi^\dagger_{,0}(y)\phi_{,0}(y)\,d^3y = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2V}\left(2a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + 2b^\dagger_{\mathbf{k}}b_{\mathbf{k}} + 0 + \ldots\right)\int d^3y = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(N_a + N_b\right) = H = P_0 \tag{6-68}$$

Thus, (6-66) can be expressed as

$$\text{only } \mu = \nu = 0 \text{ part of } \left[Q_1, Q_2^\dagger\right]_+ = \sigma^0_{12}P_0\,. \tag{6-69}$$

Now consider the case of (6-64) where $\mu = 0$ $\nu = 1$, so $\sigma^0 = I$ and $\sigma^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ (in spinor space), and the very last expression in (6-70) can be found in similar manner to that of (6-68),

$$\begin{aligned}\text{only } \mu = 0, \nu = 1 \text{ part of } \left[Q_1, Q_2^\dagger\right]_+ &= \iint \begin{pmatrix} \sigma^0_{1c}\chi_c(x)\chi^\dagger_d(y)\left(\sigma^1_{2d}\right)^\dagger \phi^\dagger_{,0}(x)\phi_{,1}(y) \\ + \chi^\dagger_d(y)\left(\sigma^1_{2d}\right)^\dagger \sigma^0_{1c}\chi_c(x)\phi_{,1}(y)\phi^\dagger_{,0}(x) \end{pmatrix} d^3x d^3y \\ &= \iint \underbrace{\left(\chi_1(x)\chi^\dagger_1(y) + \chi^\dagger_1(y)\chi_1(x)\right)}_{=\delta_{11}\delta(\mathbf{x}-\mathbf{y})}\phi^\dagger_{,0}(x)\phi_{,1}(y)\,d^3x d^3y = \int \phi^\dagger_{,0}(x)\phi_{,1}(x)\,d^3x = P_1\end{aligned} \tag{6-70}$$

(6-70) can be re-expressed as

$$\text{only } \mu = 0, \nu = 1 \text{ part of } \left[Q_1, Q_2^\dagger\right]_+ = \sigma^1_{12}P_1\,. \tag{6-71}$$

From (6-69) and (6-71), we can begin to see a pattern emerging. Hopefully, we can gain some confidence, without going through the tedium that for any values of *a* and *b*, where in each case, we sum over all μ and ν, our generalization is

$$\left[Q_a, Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab}P_\mu\,. \tag{6-72}$$

Note that (6-72) is a summation over spacetime indices, i.e., it sums energy and 3-momentum with their associated identity (for energy) and Pauli (for 3-momentum) matrices. And the particular component of each 2X2 matrix is the *a*th row and *b*th column component.

For our example where $a = 1$ and $b = 2$, (6-72) is

$$\left[Q_1, Q_2^\dagger\right]_+ = \sigma^0_{12}P_0 + \sigma^1_{12}P_1 + \sigma^2_{12}P_2 + \sigma^3_{12}P = (0)P_0 + (1)P_1 + (-i)\sigma^2_{12}P_2 + (0)P_3 = P_1 - iP_2\,. \tag{6-73}$$

Note the P_μ is the 4-momentum operator, not a number. It operates on states. If the state is an eigenstate of momentum, the operation will yield the numerical value of momentum times the original state.

$$\boxed{\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0}$$

If we were to substitute (6-55) and (6-56), along with the definition for 4-momentum in terms of the fields and their conjugate momenta (see Klauber, Vol. 1, Chaps. 3 and 4) into the LHS of (6-74) below, we could, in a lengthy procedure, deduce that it is equal to the RHS, i.e., to zero. However, there is a much simpler way to derive (6-74), but we will need a result shown in the next few pages to do so. We will do that, but for now, we will simply accept that 4-momentum commutes with Q_a and $Q_a{}^{\dagger}$, i.e.,

$$\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0\,. \qquad (6\text{-}74).$$

Note

The commutation relations can be found in another way, as shown in Aitchison, Sects. 4.1 and 4.2, pgs. 50-58. This author feels the above presentation is less opaque and more in line with the usual approach of QFT for the e/m, weak, and strong force theories.

Step #7. Determine what effect each *Qa* has on states (using the commutation relations)

Preliminary note i)

Consider a state $\left|p_\mu\, j\, m\right\rangle$, where p_μ, j, and m are the 4-momentum, total spin, and z direction spin, respectively. Then act on that state with the left side of (6-74) and $a = 1$, where primes indicate the new state arising from the action of Q_1 on the original state.

$$\begin{aligned}\left[Q_1, P_\mu\right]\left|p_\mu\, j\, m\right\rangle &= 0\left|p_\mu\, j\, m\right\rangle = 0\\ &= Q_1 P_\mu\left|p_\mu\, j\, m\right\rangle - P_\mu Q_1\left|p_\mu\, j\, m\right\rangle = Q_1 p_\mu\left|p_\mu\, j\, m\right\rangle - P_\mu\left|p'_\mu\, j'\, m'\right\rangle\\ &= p_\mu Q_1\left|p_\mu\, j\, m\right\rangle - p'_\mu\left|p'_\mu\, j'\, m'\right\rangle = p_\mu\left|p'_\mu\, j'\, m'\right\rangle - p'_\mu\left|p'_\mu\, j'\, m'\right\rangle\\ &= \left(p_\mu - p'_\mu\right)\left|p'_\mu\, j'\, m'\right\rangle \qquad \rightarrow \qquad p_\mu = p'_\mu\,.\end{aligned} \qquad (6\text{-}75)$$

The same result would be true for Q_2, $Q^\dagger{}_1$, and $Q^\dagger{}_2$.

<u>Conclusion:</u> The Q_a transformations and their complex conjugates do not change 4-momentum of a state. This is a result of the 4-momentum operator P_μ commuting with all Q_a and $Q^\dagger{}_a$.

Preliminary note ii)

Since P_μ commutes with all Q_a and $Q^\dagger{}_a$, $P^2 = P^\mu P_\mu = M^2$ also commutes with all Q_a and $Q^\dagger{}_a$,

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Homework problem 5-2: Prove P^2 commutes with all Q_a and $Q^\dagger{}_a$,

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In the following M is (rest) mass, m is spin in the z direction.

$$\begin{aligned}\left[Q_1, P^2\right]\left|p_\mu\, j\, m\right\rangle &= 0\left|p_\mu\, j\, m\right\rangle = 0\\ &= Q_1 P^2\left|p_\mu\, j\, m\right\rangle - P^2 Q_1\left|p_\mu\, j\, m\right\rangle = Q_1 M^2\left|p_\mu\, j\, m\right\rangle - P^2\left|p'_\mu\, j'\, m'\right\rangle\\ &= M^2 Q_1\left|p_\mu\, j\, m\right\rangle - M'^2\left|p'_\mu\, j'\, m'\right\rangle = M^2\left|p'_\mu\, j'\, m'\right\rangle - M'^2\left|p'_\mu\, j'\, m'\right\rangle\\ &= \left(M^2 - M'^2\right)\left|p'_\mu\, j'\, m'\right\rangle \qquad \rightarrow \qquad M^2 = M'^2\,.\end{aligned} \qquad (6\text{-}76)$$

<u>Conclusion:</u> The mass doesn't change under the operator of any of Q_a and $Q^\dagger{}_a$. This, too, is a result of the 4-momentum operator P_μ commuting with all Q_a and $Q^\dagger{}_a$. In our case, particles are massless, so their SUSY spartners are also massless.

End of preliminary notes

What does Q_1 do?

From (6-55),

$$Q_1 = \int \sigma^\nu_{1b}\chi_b(x)\phi^\dagger_{,\nu}(x)\,d^3x\,. \qquad (6\text{-}77)$$

We want to write out χ_b and $\phi^\dagger$ along with the Pauli matrices and see what Q_1 is in terms of creation and destruction opertors.

3-momentum in $+x^3$ direction

From Klauber, Vol. 2, (5-14), pg. 137, for a spinor field at the speed of light (massless field) with momentum aligned with the x^3 axis, expressed in the Weyl rep, where we note that Klauber has the LC part of the field in the top two component slots and the RC part in the bottom two.

$$\Psi_{Klauber} = \begin{bmatrix} \chi \\ \psi \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \sum_{\mathbf{p}} \sqrt{\frac{1}{V}} \left(c_1(\mathbf{p}) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-ipx} + c_2(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} + d_2^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{ipx} - d_1^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{ipx} \right) \quad (6\text{-}78)$$

Aitchison, on the other hand, whose notation we are following here, reverses the positions of the LC χ and the RC ψ.

$$\Psi = \begin{bmatrix} \psi \\ \chi \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \chi_1 \\ \chi_2 \end{bmatrix} = \sum_{\mathbf{p}} \sqrt{\frac{1}{V}} \left(c_1(\mathbf{p}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} + c_2(\mathbf{p}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-ipx} + d_2^\dagger(\mathbf{p}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{ipx} - d_1^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{ipx} \right). \quad (6\text{-}79)$$

We focus here on the LC field χ in 2D complex space to use in (6-77).

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \sum_{\mathbf{p}} \sqrt{\frac{1}{V}} \left(c_2(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ipx} - d_1^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ipx} \right) \quad (6\text{-}80)$$

In (6-80), c_2 destroys a LC particle and $d_1{}^\dagger$ creates a RC antiparticle. 3-momentum is in the $+x^3$ direction, so the particle has spin down and the antiparticle has spin up.

The complex conjugate scalar in (6-77) is written as (see Klauber, Vol. 1, (3-36), pg. 50),

$$\phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a^\dagger(\mathbf{k}) e^{ikx} + b(\mathbf{k}) e^{-ikx} \right), \quad (6\text{-}81)$$

with the derivative

$$\phi^\dagger_{,\nu}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(ik_\nu a^\dagger(\mathbf{k}) e^{ikx} - ik_\nu b(\mathbf{k}) e^{-ikx} \right) = \sum_{\mathbf{k}} \frac{ik_\nu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a^\dagger(\mathbf{k}) e^{ikx} - b(\mathbf{k}) e^{-ikx} \right). \quad (6\text{-}82)$$

Now, substitute (6-80) and (6-82) into (6-77).

$$\begin{aligned} Q_1 &= \int \sigma^\nu_{1b} \chi_b(x) \phi^\dagger_{,\nu}(x) d^3x \\ &= \int \sigma^\nu_{1b} \left(\sum_{\mathbf{p}} \sqrt{\frac{1}{V}} \left(c_2(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_b e^{-ipx} - d_1^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_b e^{ipx} \right) \right) \left(\sum_{\mathbf{k}} \frac{ik_\nu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a^\dagger(\mathbf{k}) e^{ikx} - b(\mathbf{k}) e^{-ikx} \right) \right) d^3x. \end{aligned} \quad (6\text{-}83)$$

The only components that are non-zero are for $b = 2$, so

$$Q_1 = \int \sigma^\nu_{12} \left(\sum_{\mathbf{p}} \sqrt{\frac{1}{V}} \left(c_2(\mathbf{p}) e^{-ipx} - d_1^\dagger(\mathbf{p}) e^{ipx} \right) \right) \left(\sum_{\mathbf{k}} \frac{ik_\nu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a^\dagger(\mathbf{k}) e^{ikx} - b(\mathbf{k}) e^{-ikx} \right) \right) d^3x. \quad (6\text{-}84)$$

In (6-84), from logic similar to that found in Klauber, Vol. 1, Sect. 3.4.1, pg. 53, only terms where $\mathbf{k} = \mathbf{p}$ or $\mathbf{k} = -\mathbf{p}$ survive the integration over space. We also know, from the conclusion of Preliminary note i), pg. 39, that the final state (be it the scalar or the fermion) must have the same 4-momentum as the initial state (be it the fermion or the scalar), so $\mathbf{p} = \mathbf{k}$, and we can forget about the $\mathbf{k} = -\mathbf{p}$ state. This reduces (6-84) to

$$\begin{aligned} Q_1 &= \int \sigma^\nu_{12} \left(\sum_{\mathbf{p}} \frac{ip_\nu}{\sqrt{2V\omega_{\mathbf{p}}}} \sqrt{\frac{1}{V}} \left(c_2(\mathbf{p}) a^\dagger(\mathbf{p}) e^{-ipx} e^{ipx} - d_1^\dagger(\mathbf{p}) b(\mathbf{p}) e^{ipx} e^{-ipx} \right) \right) d^3x \\ &= \sigma^\nu_{12} \sum_{\mathbf{p}} \frac{ip_\nu}{\sqrt{2E_{\mathbf{p}}}} \left(c_2(\mathbf{p}) a^\dagger(\mathbf{p}) - d_1^\dagger(\mathbf{p}) b(\mathbf{p}) \right) = \frac{i}{\sqrt{2}} \sum_{\mathbf{p}} \frac{1}{\sqrt{E_{\mathbf{p}}}} \sigma^\nu_{12} p_\nu \left(c_2(\mathbf{p}) a^\dagger(\mathbf{p}) - d_1^\dagger(\mathbf{p}) b(\mathbf{p}) \right) \quad (6\text{-}85) \\ &= \frac{i}{\sqrt{2}} \sum_{\mathbf{p}} \frac{1}{\sqrt{E_{\mathbf{p}}}} \left(\sigma^0_{12} p_0 + \sigma^1_{12} p_1 + \sigma^2_{12} p_2 + \sigma^3_{12} p_3 \right) \left(c_2(\mathbf{p}) a^\dagger(\mathbf{p}) - d_1^\dagger(\mathbf{p}) b(\mathbf{p}) \right). \end{aligned}$$

In this example, $\mathbf{p}$ is in the $+x^3$ direction, so $p_1 = p_2 = 0$, and we have

$$\begin{aligned} Q_1 &= \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left(\sigma_{12}^0 p_0 + \sigma_{12}^3 p_3\right)\left(c_2(\mathbf{p})a^\dagger(\mathbf{p}) - d_1^\dagger(\mathbf{p})b(\mathbf{p})\right) \\ &= \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left((0)p_0 + (0)p_3\right)\left(c_2(\mathbf{p})a^\dagger(\mathbf{p}) - d_1^\dagger(\mathbf{p})b(\mathbf{p})\right) = 0. \end{aligned} \tag{6-86}$$

$$Q_1\left|\mathbf{p}_L\right\rangle = 0 \quad |\mathbf{p}_L\rangle \text{ is left chiral spin down particle (or RC spin up antiparticle)} \tag{6-87}$$

Conclusion: Q_1 annihilates (turns to zero) a LC particle state with spin down

3-momentum in $-x^3$ direction

Let's try 3-momentum in the opposite direction, i.e., $\mathbf{p}$ is in the $-x^3$ direction.

In Klauber, Vol. 2, pgs. 136-137, in going from (5-11) to (5-13), we would take $p^3 \to -p^3$. So, instead of (5-14) therein ((6-78) here), we would have

$$\Psi_{Klauber} = \begin{bmatrix}\chi\\ \psi\end{bmatrix} = \begin{bmatrix}\chi_1\\ \chi_2\\ \psi_1\\ \psi_2\end{bmatrix} = \sum_{\mathbf{p}}\sqrt{\frac{1}{V}}\left(c_1(\mathbf{p})\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-ipx} + c_2(\mathbf{p})\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{-ipx} - d_2^\dagger(\mathbf{p})\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{ipx} + d_1^\dagger(\mathbf{p})\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{ipx}\right) \tag{6-88}$$

$$\Psi = \begin{bmatrix}\psi\\ \chi\end{bmatrix} = \begin{bmatrix}\psi_1\\ \psi_2\\ \chi_1\\ \chi_2\end{bmatrix} = \sum_{\mathbf{p}}\sqrt{\frac{1}{V}}\left(c_1(\mathbf{p})\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{-ipx} + c_2(\mathbf{p})\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-ipx} - d_2^\dagger(\mathbf{p})\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{ipx} + d_1^\dagger(\mathbf{p})\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{ipx}\right) \tag{6-89}$$

$$\chi = \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \sum_{\mathbf{p}}\sqrt{\frac{1}{V}}\left(c_1(\mathbf{p})\begin{pmatrix}1\\0\end{pmatrix}e^{-ipx} - d_2^\dagger(\mathbf{p})\begin{pmatrix}1\\0\end{pmatrix}e^{ipx}\right). \tag{6-90}$$

$c_1(\mathbf{p})$ destroys a LC particle and $d_1^\dagger$ creates a RC antiparticle. Since, $\mathbf{p}$ is in the $-x^3$ direction, the particle must have spin up and the antiparticle spin down.

Let's substitute (6-90) into (6-77), like we did before with (6-80) for $\mathbf{p}$ is in the $+x^3$ direction.

$$\begin{aligned} Q_1 &= \int \sigma_{1b}^\nu \chi_b(x)\phi_{,\nu}^\dagger(x)d^3x \\ &= \int \sigma_{1b}^\nu\left(\sum_{\mathbf{p}}\sqrt{\frac{1}{V}}\left(c_1(\mathbf{p})\begin{pmatrix}1\\0\end{pmatrix}_b e^{-ipx} - d_2^\dagger(\mathbf{p})\begin{pmatrix}1\\0\end{pmatrix}_b e^{ipx}\right)\right)\left(\sum_{\mathbf{k}}\frac{ik_\nu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a^\dagger(\mathbf{k})e^{ikx} - b(\mathbf{k})e^{-ikx}\right)\right)d^3x. \end{aligned} \tag{6-91}$$

The only components that are non-zero are for $b = 1$, so

$$Q_1 = \int \sigma_{11}^\nu\left(\sum_{\mathbf{p}}\sqrt{\frac{1}{V}}\left(c_1(\mathbf{p})e^{-ipx} - d_2^\dagger(\mathbf{p})e^{ipx}\right)\right)\left(\sum_{\mathbf{k}}\frac{ik_\nu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a^\dagger(\mathbf{k})e^{ikx} - b(\mathbf{k})e^{-ikx}\right)\right)d^3x. \tag{6-92}$$

Via the same logic used for (6-85), we have

$$Q_1 = \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left(\sigma_{11}^0 p_0 + \sigma_{11}^1 p_1 + \sigma_{11}^2 p_2 + \sigma_{11}^3 p_3\right)\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right). \tag{6-93}$$

For $p_1 = p_2 = 0$, this becomes, where for massless particles $p^0 = |p^3|$ and here $p^3 = -|p^3| = -p^0 = -E_{\mathbf{p}}$,

$$
\begin{aligned}
Q_1 &= \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left(p_0 + p_3\right)\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right)\\
&= \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left(p^0 - p^3\right)\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right)\\
&= \frac{i}{\sqrt{2}}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}\left(E_{\mathbf{p}} - \left(-E_{\mathbf{p}}\right)\right)\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right) = i\sqrt{2}\sum_{\mathbf{p}}\frac{p^0}{\sqrt{E_{\mathbf{p}}}}\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right)\\
&= i\sqrt{2}\sum_{\mathbf{p}}\sqrt{E_{\mathbf{p}}}\left(c_1(\mathbf{p})a^\dagger(\mathbf{p}) - d_2^\dagger(\mathbf{p})b(\mathbf{p})\right).
\end{aligned}
\tag{6-94}
$$

(6-94) is not zero, unlike (6-86), so the creation and destruction operators will get a chance to operate on states. Looking at the construction and destruction operators we can conclude as follows.

<u>Conclusion #1</u>: Q_1 destroys an LC particle with spin up (and 3-momentum down) via c_1 and creates a scalar particle via $a^\dagger$. Q_1 also destroys an anti-scalar particle via b and creates an RC antiparticle with spin down (and 3-momentum down) via $d_2^\dagger$.
<u>Conclusion #2</u>: In the process, Q_1 leaves state 4-momentum unchanged **($\mathbf{p} \rightarrow \mathbf{p}$)**.

<u>Conclusion #3</u>: Q_1 annihilates all other states (transforms them to zero).

As an aside, recall from Wholeness Chart 6-2, pg. 26 herein and Klauber, Vol. 2, Sect. 2.6, pgs. 45-47, that the spin matrix operating on the field Ψ gives the same eigenvalues as the spin operator for states does acting on a single particle fermion state $|\Psi\rangle$. So, if we operated on the field (6-90) with the field spin operator in the Weyl rep (see Klauber, Vol. 2, Chap. 5, (5-20), pg. 139, repeated below as the LH of (6-95)), we find it has spin up, which is what the related fermion particle state has, and which we deduced in other ways above.

$$
\text{Weyl rep, 4D spinor space}\quad {}_W\Sigma_3 = \frac{1}{2}\begin{bmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1\end{bmatrix}\qquad\qquad \text{2D LC spinor space}\quad {}_W\sigma_3 = \frac{1}{2}\begin{bmatrix}1 & \\ & -1\end{bmatrix}. \tag{6-95}
$$

See Wholeness Chart 6-1, pg. 25, for a summary of the above analysis of Q_1 and the following.

What does $Q_1^\dagger$ do?

If we went through all of the math, we would find $Q_1^\dagger$ is the inverse of Q_1. It reverses the action of Q_1.

<u>Conclusion</u>: $Q_1^\dagger$ creates an LC particle with spin up (and 3-momentum down) and destroys a scalar particle. $Q_1^\dagger$ also creates an anti-scalar particle and destroys an RC antiparticle with spin down (and 3-momentum down).
<u>Conclusion #2</u>: It annihilates all other states (transforms them to zero).

What does Q_2 do?

Going through all the steps like we have done above for Q_1 for Q_2, we would find the following.

<u>Conclusion #1</u>: Q_2 destroys an LC particle with spin down (and 3-momentum up) and creates a scalar particle. Q_2 also destroys an anti-scalar particle and creates an RC antiparticle with spin up (and 3-momentum up).
<u>Conclusion #2</u>: Q_2 annihilates all other states (transforms them to zero).

What does $Q_2^\dagger$ do?

Going through all the math, we would find $Q_2^\dagger$ is the inverse of Q_2. It reverses the action of Q_2.

<u>Conclusion #1</u>: $Q_2^\dagger$ creates an LC particle with spin down (and 3-momentum up) and destroys a scalar particle. $Q_2^\dagger$ also creates an anti-scalar particle and destroys an RC antiparticle with spin up (and 3-momentum up).
<u>Conclusion #2</u>: It annihilates all other states (transforms them to zero).

Note #1: In essence, Q_1 and Q_2 do the same thing. They change an LC fermion into a scalar. The only difference in the two operators is that the first lowers a spin $m = +½$ to a spin 0, whereas the second raises a spin $m = -½$ to a spin 0. But, in both cases the magnitude of the initial spin $j = ½$ is the same, and the chirality/helicity (LC=LH) is the same. The difference is only relative to the particular alignment of the x^3 axis we chose to employ. Physically, there is no difference between the effects of Q_1 and Q_2.

This is the main reason, I believe, why we call this $N = 1$ SUSY. Ultimately, though we work with two transformations (and their complex conjugate transposes), there is really only one physical effect.

Note #2: The action of the Q_a on states can be also be found using the commutation relations (6-9) and a relation between the spin operator and the Q_a, as in Aitchison, Sect. 4.4, pg. 61. However, the approach herein more closely parallels the usual approach in QFT to the e/m, weak, and strong force theories, plus shows clearly how the antiparticles come into play.

6.7.2 Right Chiral Fermions and Left Chiral Antifermions

All of the prior sub-section was done for LC fermions (and RC antifermions) and their SUSY spartner scalars. These form weak interaction doublets in $SU(2)$ space.

RC fermions (and LC antifermions) are a separate animal, and are singlets in weak $SU(2)$ space. However, both LC fermions and RC fermions form doublets in superspace with their SUSY spartners, as do their antiparticles.

We could derive the SUSY charge operators for RC fermions (and LC antifermions) in the same manner as we did in Sect. 6.7.1 for LC fermions (and RC antifermions). That is, we could follow the same 7 steps we did there, but we won't do it here. We do, however, show the results of this in the bottom half of Wholeness Chart 6-5, pg. 29.

6.7.3 The Proof We Skipped Over Earlier

Homework Problem 6-1.: We are now prepared to prove (6-74), and we will do it as homework. That is, show that 4-momentum and Q_a commute, as do 4-momentum and $Q_a^\dagger$, i.e., show $\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0$. Hint: Operate on a state and note that, as shown above, 4-momentum is unchanged under the action of Q_a and $Q_a^\dagger$.

6.7.4 Summary of the Commutation Relations

Note that the key anti-commutators and commutators (for N=1 SUSY) are summarized in (6-9).

6.8 Solutions to Homework Problems

Homework Problem 6-1.: Show (6-74), i.e., $\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0$.

Ans. Consider p^μ and s as the 4-momentum and spin of the state $|p^\mu, s\rangle$ and look first at Q_1. The eigenvalue of the 4-momentum operator P^μ is the number p^μ.

$$\begin{aligned}
\left[Q_1, P^\mu\right]\left|p^\mu s\right\rangle &= Q_1 P^\mu \left|p^\mu s\right\rangle - P^\mu Q_1 \left|p^\mu s\right\rangle = Q_1 p^\mu \left|p^\mu s\right\rangle - P^\mu \left|p^\mu, s-\tfrac{1}{2}\right\rangle \quad \left(\text{Note: } p^\mu \text{ unchanged under } Q_1\right) \\
&= p^\mu Q_1 \left|p^\mu s\right\rangle - p^\mu \left|p^\mu, s-\tfrac{1}{2}\right\rangle = p^\mu \left|p^\mu, s-\tfrac{1}{2}\right\rangle - p^\mu \left|p^\mu, s-\tfrac{1}{2}\right\rangle = 0 \\
\left[Q_2, P^\mu\right]\left|p^\mu s\right\rangle &= Q_2 P^\mu \left|p^\mu s\right\rangle - P^\mu Q_2 \left|p^\mu s\right\rangle = Q_2 p^\mu \left|p^\mu s\right\rangle - P^\mu \left|p^\mu, s+\tfrac{1}{2}\right\rangle \\
&= p^\mu Q_2 \left|p^\mu s\right\rangle - p^\mu \left|p^\mu, s+\tfrac{1}{2}\right\rangle = p^\mu \left|p^\mu, s+\tfrac{1}{2}\right\rangle - p^\mu \left|p^\mu, s+\tfrac{1}{2}\right\rangle = 0
\end{aligned} \qquad (6\text{-}96)$$

This result is true for any fermion state (different type or momentum), so therefore (6-74) is true. Similar analysis holds for $Q_a^\dagger$.

In (6-75), we assumed (6-74) was true to prove Q_a doesn't change the 4-momentum of a state. Here, we did the reverse. We know (which we didn't when (6-74) was introduced) that 4-momentum is unchanged under the action of Q_a. (See (6-94) and Conclusion #2 following it.) Using that in (6-96), we showed that (unchanging 4-momentum) means Q_a commutes with P^μ, i.e., (6-74).

Homework problem 5-2: Prove P^2 commutes with all Q_a and $Q^\dagger{}_a$,

Ans.

$$\begin{aligned}
\left[Q_a, P^2\right] &= Q_a P^\mu P_\mu - P^\mu P_\mu Q_a = P^\mu Q_a P_\mu - P^\mu Q_a P_\mu = 0 \\
\left[Q_a^\dagger, P^2\right] &= Q_a^\dagger P^\mu P_\mu - P^\mu P_\mu Q_a^\dagger = P^\mu Q_a^\dagger P_\mu - P^\mu Q_a^\dagger P_\mu = 0
\end{aligned} \qquad (6\text{-}97)$$

7 The Wess-Zumino SUSY Model

7.1 The Wess-Zumino Lagrangian

7.1.1 Stating It

In this sub-section we merely summarize the Wess-Zumino model Lagrangian. The next sub-section discusses how it is derived.

As in the prior chapter in Aitchison and the prior section herein, we focus on the LC field χ, and for the moment, ignore the RC field ψ. The subscript *WZ* does *not* refer to W and Z fields in weak interactions, but to Wess and Zumino, the discoverers of this model.

$$\mathcal{L}_{\text{free WZ}} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i \bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i \qquad i = \text{flavor (or later, gauge)} \qquad \text{Aitchison (5.1) [70]} \qquad (7\text{-}1)$$

For the interaction Lagrangian (for ϕ, χ, and F type fields), we will need a quantity called the superpotential, symbolized by W, where y_{ijk} are called the Yukawa couplings.

$$W = \tfrac{1}{2} M_{ij} \phi_i \phi_j + \tfrac{1}{6} y_{ijk} \phi_i \phi_j \phi_k \qquad \text{Aitchison (5.9) [72]} \qquad (7\text{-}2)$$

From (7-2), we define

$$W_i = \frac{\partial W}{\partial \phi_i} = M_{ij} \phi_j + \tfrac{1}{2} y_{ijk} \phi_j \phi_k \qquad \text{Aitchison (5.17) [73]} \qquad (7\text{-}3)$$

$$W_{ij} = \frac{\partial^2 W}{\partial \phi \partial \phi_j} = M_{ij} + y_{ijk} \phi_k \qquad \text{Aitchison (5.8) [72].} \qquad (7\text{-}4)$$

Then, the interaction Lagrangian is (where *h.c.* means Hermitian conjugate of prior term(s))

$$\mathcal{L}_{\text{int WZ}} = -\left|W_i\right|^2 - \tfrac{1}{2}\left\{W_{ij} \chi_j \cdot \chi_j + h.c.\right\} \qquad \text{See Aitchison (5.22) [74],} \qquad (7\text{-}5)$$

and the total Lagrangian (for these particular fields) is

$$\mathcal{L}_{WZ} = \mathcal{L}_{\text{free WZ}} + \mathcal{L}_{\text{int WZ}} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i \bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i - \left|W_i\right|^2 - \tfrac{1}{2}\left\{W_{ij} \chi_j \cdot \chi_j + h.c.\right\}. \qquad (7\text{-}6)$$

7.1.2 Deriving it

Each term in the Lagrangian has dimension 4. By demanding that interactions are renormalizable, the coupling constants have to be dimensionless or have negative dimensions. See Klauber, Vol. 2, Sect. 16.3.2, pgs. 475-476, with Bottom Line summary on pg. 476.

From this restriction and one other on ϕ_i (see Aitchison, pg. 71) that is needed for an invariant action, Aitchison (Chap. 5) deduces the most general form of $\mathcal{L}_{\text{int WZ}}$ to be as in (7-7). At this point W_i and W_{ij} are unknown functions of the scalar fields ϕ_i, and the shorthand notation for the argument (ϕ, $\phi^\dagger$) represents all scalar fields ϕ_i and their complex conjugates

$$\mathcal{L}_{\text{int WZ}} = W_i\left(\phi, \phi^\dagger\right) F_i - \tfrac{1}{2} W_{ij}\left(\phi, \phi^\dagger\right) \chi_i \cdot \chi_j + h.c. \qquad \text{Aitchison (5.3) [71]} \qquad (7\text{-}7)$$

By extending the criterion of constraining the Lagrangian to result in an invariant action, (Aitchison, pgs. 71-73), from (7-7) one can deduce (7-5) as the only possible form for the interaction Lagrangian for these fields.

Bottom line: We get the specific form for the interaction Lagrangian for scalar, LC fermion, and auxiliary F type fields (7-5) by requiring

1) renormalization (so all coupling constants have dimension ≤ 0) and
2) an invariant action under the SUSY transformation set (6-26) to (6-28).

7.2 Special Case Example: A Single Chiral Superfield

To simplify, in the following, we will only consider a single χ and its single spartner ϕ. That is i = only 1 in (7-6), and we ignore the subscript i.

7.2.1 Masses of Fermions and Sfermions

Using (7-6) for a single χ and a single ϕ in the Euler-Lagrange equation, Aitchison (pgs. 74-76) shows that the equations of motion for the scalar and fermion fields are

$$\partial_\mu \partial^\mu \phi + |M|^2 \phi = 0 \qquad \text{Aitchison (5.29) [75]} \qquad (7\text{-}8)$$

$$\partial_\mu \partial^\mu \chi + |M|^2 \chi = 0 \qquad \text{Aitchison (5.41) [76].} \qquad (7\text{-}9)$$

This confirms that under the SUSY transformation between χ and ϕ, the mass is unchanged, which we already showed in (6-76) and summarized in the conclusion below that equation.

So far, for us, we've dealt with massless particles, for both fermions and their scalar spartners. However, we know that in our contemporary universe, after Higgs symmetry breaking, we can't have partners and spartners with the same mass, since no spartners are seen at the standard model particle mass levels. So, there must be some symmetry breaking in SUSY (as we already know anyway) in which the spartners obtain masses that are much greater than their SM partners. This is an advanced topic, and we won't delve into it here.

7.2.2 Interactions

For a single fermion/sfermion pair, (7-6), with (7-3) and (7-4) becomes

$$\begin{aligned}
\mathcal{L}_{WZ} &= \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F - |W_1|^2 - \tfrac{1}{2}\{W_{11}\chi \cdot \chi + h.c.\} \\
&= \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F - \left|M\phi + \tfrac{1}{2} y\phi^2\right|^2 - \tfrac{1}{2}\{(M + y\phi)\chi \cdot \chi + h.c.\} \\
&= \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F - |M|^2 \phi\phi^\dagger - \tfrac{1}{2}\left(M\, y^* \phi\phi^{\dagger 2} + M^* y \phi^2 \phi^\dagger\right) - \tfrac{1}{4}|y|^2 \phi^2 \phi^{\dagger 2} \\
&\qquad - \tfrac{1}{2}\{M\chi \cdot \chi + h.c.\} - \tfrac{1}{2}\{y\phi\chi \cdot \chi + h.c.\}.
\end{aligned} \qquad (7\text{-}10)$$

We can gather terms in (7-10) of the same order into groups.

Free (Quadratic) Terms

$$\mathcal{L}_{WZ\,quad} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F - |M|^2 \phi\phi^\dagger - \tfrac{1}{2}\{M\chi \cdot \chi + h.c.\} \qquad (7\text{-}11)$$

Interaction Terms

Interaction terms will all show up as vertices in Feynman diagrams, of three or four particles, which we break into the following categories. (7-12) below will represent vertices for three scalars; (7-13) for four scalars; and (7-14) for two fermions and a scalar.

$$\mathcal{L}_{WZ\,cubic} = -\tfrac{1}{2}\left(M\, y^* \phi\phi^{\dagger 2} + M^* y\, \phi^2 \phi^\dagger\right) \qquad (7\text{-}12)$$

$$\mathcal{L}_{WZ\,quartic} = -\tfrac{1}{4}|y|^2 \phi^2 \phi^{\dagger 2} \qquad (7\text{-}13)$$

$$\mathcal{L}_{WZ\,Yukawa} = -\tfrac{1}{2}\{y\phi\chi \cdot \chi + h.c.\} \qquad (7\text{-}14)$$

So,

$$\mathcal{L}_{WZ} = \underbrace{\mathcal{L}_{WZ\,quad}}_{\text{free}} + \underbrace{\mathcal{L}_{WZ\,cubic} + \mathcal{L}_{WZ\,quaritc} + \mathcal{L}_{WZ\,Yukawa}}_{\text{intereaction}}. \qquad (7\text{-}15)$$

7.3 Converting to Majorana Rep

In the section after this one, we will need to have the fermion fields in the *WZ* Lagrangian in the Majorana rep and the complex scalar ϕ expressed in terms of the real fields A and B as

$$\phi = \frac{1}{\sqrt{2}}(A + iB). \qquad (7\text{-}16)$$

Since the fermion fields in the Majorana rep are real, using them and the fields A and B in place of ϕ provides us with relations in terms of purely real fields.

In Klauber, Vol. 2, Chap. 5, (5-9), pg. 136, we find the transformation for spinor fields from the standard rep to the Weyl rep ${}_{S\text{-}W}U$. Being unitary, its inverse, from the Weyl rep to the standard rep is simply

$$ {}_{W\to S}U = {}_{S\to W}U^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \qquad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \quad \text{Klauber notation} \qquad (7\text{-}17) $$

In Aitchison's notation, the top two rows of a 4-spinor are exchanged with the bottom two rows. So, the matrix is changed by interchanging the first and second columns.

$$ {}_{W\to S}U = {}_{S\to W}U^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \quad \text{Aitchison notation.} \qquad (7\text{-}18) $$

In the same reference, (5-24) takes spinor fields from the standard to the Majorana rep.

$$ {}_{S\to M}U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & \sigma_2 \\ \sigma_2 & -I \end{bmatrix} \qquad \text{Klauber notation.} \qquad (7\text{-}19) $$

$$ {}_{S\to M}U = \frac{1}{\sqrt{2}}\begin{bmatrix} \sigma_2 & I \\ -I & \sigma_2 \end{bmatrix}. \qquad \text{Aitchison notation} \qquad (7\text{-}20) $$

So, to transform the Weyl spinor we have been using throughout to the Majorana rep, we simply transform it first to the standard rep via (7-18) and then that result to the Majorana rep via (7-20).

$$ {}_{S\to M}U\,{}_{W\to S}U\begin{bmatrix} 0 \\ \chi \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sigma_2 & I \\ -I & \sigma_2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} 0 \\ \chi \end{bmatrix} = \Psi_M^{\chi} \quad \left(\text{LC Weyl spinor in Majorana rep}\right) \qquad (7\text{-}21) $$

Note that $\bar{\Psi}_M^{\chi}$ generally has four non-zero components, whereas the 4-spinor for a LC fermion in the Weyl rep has only the bottom 2 components as non-zero. Note also the following identities, which, given time, one could work out using the appropriate transformations, and which we will use below to get (7-23) and (7-26).

$$ \chi^{\dagger} i\bar{\sigma}^{\mu}\partial_{\mu}\chi \xrightarrow[\text{rep}]{\text{to Majorana}} \bar{\Psi}_M^{\chi} i\gamma^{\mu}\partial_{\mu}\Psi_M^{\chi} $$

$$ \chi\cdot\chi + h.c. = \chi\cdot\chi + \chi^{\dagger}\cdot\chi^{\dagger} \xrightarrow[\text{rep}]{\text{to Majorana}} \Psi_M^{\chi}\bar{\Psi}_M^{\chi} \qquad \chi\cdot\chi - \chi^{\dagger}\cdot\chi^{\dagger} \xrightarrow[\text{rep}]{\text{to Majorana}} \Psi_M^{\chi}\gamma^5\bar{\Psi}_M^{\chi} \qquad .(7\text{-}22) $$

Thus, (7-11) to (7-14) become, in the Majorana rep, where we assume M and y are real numbers,

$$ \begin{aligned} \mathcal{L}_{WZ\,quad} &= \chi^{\dagger} i\bar{\sigma}^{\mu}\partial_{\mu}\chi - \tfrac{1}{2}\left\{M\,\chi\cdot\chi + \chi^{\dagger}\cdot\chi^{\dagger}\right\} + \partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi - |M|^2\,\phi\phi^{\dagger} + F^{\dagger}F \\ &= \tfrac{1}{2}\bar{\Psi}_M^{\chi}\left(i\gamma^{\mu}\partial_{\mu} - M\right)\Psi_M^{\chi} + \left(\partial_{\mu}\tfrac{1}{\sqrt{2}}(A - iB)\right)\partial^{\mu}\tfrac{1}{\sqrt{2}}(A + iB) - |M|^2\tfrac{1}{\sqrt{2}}(A + iB)\tfrac{1}{\sqrt{2}}(A - iB) + F^{\dagger}F \qquad (7\text{-}23) \\ &= \tfrac{1}{2}\bar{\Psi}_M^{\chi}\left(i\gamma^{\mu}\partial_{\mu} - M\right)\Psi_M^{\chi} + \tfrac{1}{2}\partial^{\mu}A\,\partial_{\mu}A + \tfrac{1}{2}\partial^{\mu}B\,\partial_{\mu}B - \tfrac{1}{2}M^2A^2 - \tfrac{1}{2}M^2B^2 + F^{\dagger}F \end{aligned} $$

$$ \begin{aligned} \mathcal{L}_{WZ\,cubic} &= -\tfrac{1}{2}M\,y\tfrac{1}{\sqrt{2}}(A + iB)\left(\tfrac{1}{\sqrt{2}}(A - iB)\right)^2 - \tfrac{1}{2}M^{*}y\left(\tfrac{1}{\sqrt{2}}(A + iB)\right)^2\left(\tfrac{1}{\sqrt{2}}(A - iB)\right) \\ &= -\tfrac{1}{2}M\frac{y}{2\sqrt{2}}\Big(A + \underbrace{iB}_{\text{cancels}}\Big)\left(A^2 + B^2\right) - \tfrac{1}{2}M\frac{y}{2\sqrt{2}}\left(A^2 + B^2\right)\Big(A - \underbrace{iB}_{\text{cancels}}\Big) = -M\frac{y}{2\sqrt{2}}A\left(A^2 + B^2\right) \qquad (7\text{-}24) \\ &= -M\,gA\left(A^2 + B^2\right) \qquad \text{where } g = \frac{y}{2\sqrt{2}} \end{aligned} $$

$$ \begin{aligned} \mathcal{L}_{WZ\,quartic} &= -\tfrac{1}{4}y^2\left(\tfrac{1}{\sqrt{2}}(A + iB)\right)^2\left(\tfrac{1}{\sqrt{2}}(A - iB)\right)^2 = -\tfrac{1}{16}y^2(A + iB)(A - iB)(A + iB)(A - iB) \\ &= -\tfrac{1}{16}y^2\left(A^2 + B^2\right)\left(A^2 + B^2\right) \qquad (7\text{-}25) \\ &= -\tfrac{1}{2}g^2\left(A^2 + B^2\right)^2 \qquad \text{where } g = \frac{y}{2\sqrt{2}} \end{aligned} $$

$$
\begin{aligned}
\mathcal{L}_{WZ\,Yukawa} &= -\tfrac{1}{2}\{y\phi\chi\cdot\chi + h.c.\} = -\tfrac{y}{2}\left\{\left(\tfrac{1}{\sqrt{2}}(A+iB)\right)\chi\cdot\chi + \left(\tfrac{1}{\sqrt{2}}(A-iB)\right)(\chi\cdot\chi)^{\dagger}\right\} \\
&= -\tfrac{y}{2\sqrt{2}}\left\{A\chi\cdot\chi + iB\chi\cdot\chi + A(\chi\cdot\chi)^{\dagger} - iB(\chi\cdot\chi)^{\dagger}\right\} = -\tfrac{y}{2\sqrt{2}}\left\{A\left(\chi\cdot\chi + (\chi\cdot\chi)^{\dagger}\right) + iB\left(\chi\cdot\chi - (\chi\cdot\chi)^{\dagger}\right)\right\} \\
&= -\tfrac{y}{2\sqrt{2}}\left\{A\Psi^{\chi}_{M}\bar{\Psi}^{\chi}_{M} + iB\Psi^{\chi}_{M}\gamma^{5}\bar{\Psi}^{\chi}_{M}\right\} \\
&= -g\left\{A\Psi^{\chi}_{M}\bar{\Psi}^{\chi}_{M} + iB\Psi^{\chi}_{M}\gamma^{5}\bar{\Psi}^{\chi}_{M}\right\} \qquad \text{where } g = \tfrac{y}{2\sqrt{2}}.
\end{aligned}
\tag{7-26}
$$

7.4 Cancellation of Divergences in the Wess-Zumino Model

7.4.1 Introduction

Recall from Sect. 2.2.1 pg. 6, on the Higgs gauge hierarchy problem, the main contributions to the quadratic divergences in the Higgs mass using cut-off regularization, shown there in Figs. 2, 3, and 4 and added together in (2-14). In Sect. 2.4, pg. 10, we outlined how SUSY might eliminate these divergences.

We now show explicitly how, in the Wess-Zumino model, they are eliminated. Or, at least, we will show how the contributions from the scalar Higgs and fermion loops of Figs. 2 and 3 can be cancelled. We will here ignore the W and Z loop contributions, as we have yet to examine SUSY for spin 1 fields, though one can expect similar SUSY effects for them.

As noted in Sect. 2, the Higgs mass, represented as an **X** in Feynman diagrams, has higher order loop corrections that modify the value of that mass. These many loops are all represented collectively in Fig. 7-1 in the commonly used symbol of a grey circle. In the prior section we represented the Majorana SUSY scalars by the letters A and B. We show only one of these in Fig. 7-1, and treat only that one in the analysis following. Similar results follow directly for the SUSY scalar B. We assume A represents the Higgs field/particle.

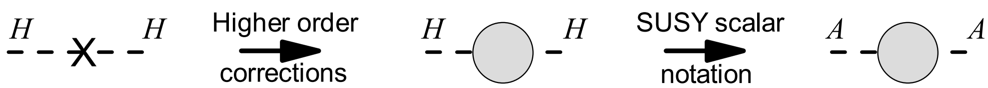

Figure 7-1. Corrections to the Higgs Mass from Higher Order Loops

Assuming the incoming and outgoing Higgs particles are virtual with 4-momentum $\hat{k}$, we have an amplitude for the corrections in Fig. 7-1 of

$$
\mathcal{M} = i\Delta_F(\hat{k})\, i\Pi\, i\Delta_F(\hat{k}) = \frac{i}{\hat{k}^2 - M^2}\, i\Pi\, \frac{i}{\hat{k}^2 - M^2} \qquad i\Pi = \underbrace{i\Pi_1}_{\text{1st loop}} + \underbrace{i\Pi_2}_{\text{2nd loop}} + \dots\,, \tag{7-27}
$$

where each $i\Pi_n$ represents a different internal loop (or set of loops) diagram. The $i\Delta_F$ propagators are the same for every diagram, so we only need to examine $i\Pi$.

The scalar diagrams that give us quadratic divergences, via the cut-off regularization method, are those of Sect. 2, Figs. 2 and 3. We now show that SUSY can cancel the contributions from those diagrams. We will first present the amplitude for each figure, then in a subsequent sub-section, outline how those are arrived at.

7.4.2 Scalar Loops

All quadratically divergent loops (via cut-off regularization) with scalars are shown in Fig. 7-2. These also include log divergences, but those are not critical, so we focus on the quadratic parts here.

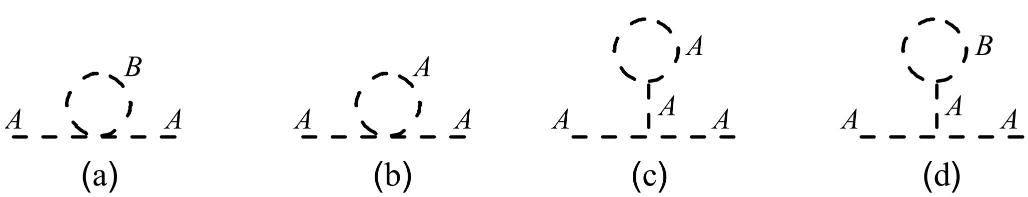

Figure. 7-2. Quadratic Scalar Loops
(In Aitchison, Figs. 5.1, 5.2, 5.3, and 5.4, pgs. 79-81)

For Fig. 7-2(a)

Let's look first at the contribution to $i\Pi$ from Fig. 7-2(a). Aitchison deduces this using the Green function approach on pgs. 78-79 to find his (5.56). The reader should study that, but we will do it here in an alternative way, with the canonical method.

Recall from Klauber, Vol. 1, Wholeness Chart 8-4, pg. 249, the interaction amplitude between an initial state i and a final state f is

$$S_{fi} = \langle f|S|i\rangle , \tag{7-28}$$

where S is the time ordered operator found from the interaction Lagrangian in the interaction picture $\mathcal{L}_I^I = -\mathcal{H}_I^I$,

$$S = \underbrace{I}_{S^{(0)}} \underbrace{-i\int_{-\infty}^{\infty}\mathcal{H}_I^I\left(x_1\right)d^4x_1}_{S^{(1)}} \underbrace{-\tfrac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\mathcal{H}_I^I\left(x_1\right)\mathcal{H}_I^I\left(x_2\right)\right\}d^4x_1d^4x_2}_{S^{(2)}} + \ldots\ldots \tag{7-29}$$

We evaluate (7-29) using Wick's theorem for time ordered operators $A,B,C,\ldots$,

$$\begin{aligned} T\left\{(AB\ldots)_{x_1}\ldots\ldots(AB\ldots)_{x_n}\right\} &= N\left\{(AB\ldots)_{x_1}\ldots\ldots(AB\ldots)_{x_n}\right\} \\ &+N\left\{(AB..)_{x_1}(AB..)_{x_2}\ \ldots\right\}+N\left\{(AB..)_{x_1}(AB..)_{x_2}\ ..\right\}+\ldots+N\left\{(ABC\ldots)_{x_1}(ABC\ldots)_{x_2}\ \ldots\right\}+\ldots \end{aligned} \tag{7-30}$$

$A,B,C, \ldots$ represent quantum fields, such as ψ, $\bar{\psi}$, A^μ in QED, and here, the A and B of Fig. 7-2. With (7-30) in (7-29), and that S in (7-28), we get the amplitude S_{fi}.

For 7-2(a), we consider the incoming and outgoing A fields to be propagators, in order to keep things simplest. We can imagine one scenario in which two incoming real particles annihilate one another, and an A is created. Similarly, the outgoing A, at a vertex, turns into outgoing real particles. There are, of course, many ways to create an incoming propagator and an outgoing propagator, but in order to simplify, we focus on just the propagators themselves.

We know from (7-25) that

$$\mathcal{H}_{I\,WZ\,quartic}^{I} = -\mathcal{L}_{WZ\,quartic} = \tfrac{1}{2}g^2\left(A^2+B^2\right)^2 = \tfrac{1}{2}g^2A^4 + g^2A^2B^2 + \tfrac{1}{2}g^2B^4 . \tag{7-31}$$

The next to last term in (7-31) gives rise to the vertex at the center of Fig. 7-2(a). Other terms will give rise to the incoming and outgoing A particles (propagators). If x is the incoming spacetime value (where we streamline by taking $x = x^\mu$), y is the outgoing value, and z is the value at the four-vertex, then, we will find terms in (7-30), and thus, (7-29), of two forms,

$$A(x)\ \ A(z)A(z)B(z)B(z)\ \ A(y) \qquad\qquad A(x)\ \ A(z)A(z)B(z)B(z)\ \ A(y)\ . \tag{7-32}$$

Each of the two terms in (7-32) must be counted equally in the determination of the total amplitude, and for our purposes for the determination of the loop factor $i\Pi_i$ of (7-27), where i here means 7-2(a), and which we designate as $-i\Pi_A^{(B)}$.

Recall from Klauber, Vol. 1, Wholeness Chart 5-4, pg. 160, that the scalar propagator in momentum space is

$$\Delta_F\left(k\right) = \frac{1}{k^2-\mu^2+i\varepsilon} \qquad \text{mass } \mu = M \text{ in Aitchison ,} \tag{7-33}$$

and this is the form of the propagators for the B field loop in (7-32).

So, with the Dyson-Wicks expansion in (7-29) we get

$$S_A^{(B)} \propto -g^2\iiint N\Big(A(x)A(z)A(z)B(z)B(z)A(y)\Big)+N\Big(A(x)A(z)A(z)B(z)B(z)A(y)\Big)d^4xd^4yd^4z \tag{7-34}$$

We can exchange dummy variables x and y in the second term of (7-34) to get

$$
\begin{aligned}
S_A^{(B)} &= -g^2 \iiint N\Big(A(x)A(z)A(z)B(z)B(z)A(y) \Big) + N\Big(A(y)A(z)A(z)B(z)B(z)A(x) \Big) d^4x d^4y d^4z \\
&= -2g^2 \iiint N\Big(A(x)A(z)A(z)B(z)B(z)A(y) \Big) d^4x d^4y d^4z \qquad \text{(7-35)} \\
&= -2g^2 \iiint i\Delta_F(x-z)\, i\Delta_F(z-z)\, i\Delta_F(y-z)\, d^4x d^4y d^4z.
\end{aligned}
$$

Our Feynman amplitude in momentum space thus becomes (compare with (7-27)), where we can read off (7-37)

$$
\mathscr{M} = i\Delta_F(\hat{k})\, i\Pi_A^{(B)}\, i\Delta_F(\hat{k}) = \frac{i}{\hat{k}^2 - M^2} \underbrace{\left(-2g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \right)}_{i\Pi_A^{(B)}} \frac{i}{\hat{k}^2 - M^2}. \qquad \text{(7-36)}
$$

where M is the mass of the B particle (as well as of the A particle), and we have to integrate the loop over all k. Thus,

$$
\text{Fig. 7-2(a)} \qquad -i\Pi_A^{(B)} = 2g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.56) [79],} \qquad \text{(7-37)}
$$

Although we won't be needing it what follows, using cut-off regularization, Aitchison deduces

$$
\begin{aligned}
-i\Pi_A^{(B)} &= 2g^2 \frac{1}{8\pi^2} \left(\Lambda^2 - M^2 \ln(\Lambda / M) + \text{ finite terms as } \Lambda \to \infty \right) \quad \text{" (5.61) [80]} \\
&= \underbrace{2g^2 \frac{1}{8\pi^2} \Lambda^2}_{\text{quadratic part}} + \underbrace{\ldots}_{\substack{\text{lower order} \\ \text{and finite parts}}} , \qquad \text{(7-38)}
\end{aligned}
$$

which has quadratic divergence in energy level Λ.

<u>For Fig. 7-2(b)</u>

Homework Problem 7-1. Derive (7-39).

<u>Other Diagrams in Fig. 7-2</u>

One can deduce the other quadratic contributions shown in Fig. 7-2. Aitchison shows these deviations in abbreviated form on pgs. 78-85.

$$
\text{Fig. 7-2(b)} \qquad -i\Pi_A^{(A)} = 6g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.64) [80]} \qquad \text{(7-39)}
$$

$$
\text{Fig. 7-2(c)} \qquad -i\Pi_A^{(t,A)} = -18g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.69) [82]} \qquad \text{(7-40)}
$$

$$
\text{Fig. 7-2(d)} \qquad -i\Pi_A^{(t,B)} = -6g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.72) [83]} \qquad \text{(7-41)}
$$

7.4.3 Fermion Loops

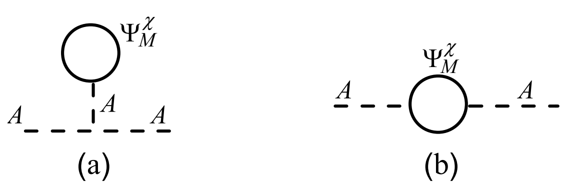

Figure . 7-3. Quadratic Fermion Loops
(In Aitchison, Figs. 5.5 and 5.8, pgs. 82,84)

We state without proof that

$$\text{Fig. 7-3(a)} \qquad -i\Pi_A^{(t,\chi)} = 24g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.74) [83]} \qquad (7\text{-}42)$$

$$\text{Fig. 7-3(b)} \qquad -i\Pi_A^{(\chi\, quad\ loop)} = -8g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \qquad \text{Aitchison (5.84) [85]} \qquad (7\text{-}43)$$

7.4.4 Adding All Loops Together

$$-i\Pi_A = (2 + 6 - 18 - 6 + 24 - 8)\, g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} = 0 \qquad (7\text{-}44)$$

If A is the Higgs scalar, then the quadratic divergences to its mass sum to zero, assuming it has a sfermion SUSY partner χ.

Special Note on Fig. 7-3(a).

You may recall studying tadpole diagrams, such as shown in Fig. 7-3(a), where for fermion loops the contribution to the amplitude is zero. This is because, for every fermion loop (with arrows in one direction) there is an antifermion loop (with arrows in the opposite direction). Antifermion propagators have the same magnitude, but opposite sign, from their sibling fermion propagators. Each contributes to the amplitude, but they cancel one another leaving a net contribution of zero.

The difference from the usual tadpole diagrams with fermion loops is that here we have a Majorana fermion (sfermion, to be precise), which is its own antiparticle. So, there is no second diagram to cancel the first one, and the sfermion tadpole diagram does make a contribution to the amplitude, which turns out to be (7-42).

7.5 Final Note

As we have noted above and in earlier parts of this document, the quadratic divergences with Λ occur for the cut-off method of regularization, but not the dimensional analysis (DR) method. So, it seems legitimate to question whether there really is an issue with the mass of the Higgs, since the cut-off method did not work in QED, but the DR method did.

Martin[2] notes, however, that with DR, one finds a quadratic divergence with the mass of any heavy complex scalar (which may exist but we have not detected) coupled to the Higgs. So, the problem still exists there. The SUSY solution shown above eliminates these contributions, however, before we choose either the cut-off or DR method to evaluate the integral in the first line of (7-37), which recurs for all other contributions (7-39) to (7-43).

7.6 Solution to Homework Problem

Homework Problem 7-1. Derive (7-39)

Ans. Fig. 7-2(b) arises from the first term after the last equal sign in (7-31), $\frac{1}{2}g^2A^4$. Its analysis follows what we did above for Fig. 7-2(a) and the second term after the last equal sign in (7-31), except for the factor of ½ and the different number of separate subamplitudes from (7-32). The first four of the subamplitudes have contractions like this.

$$A(x)\;\; A(z)A(z)A(z)A(z)\;\; A(y) \qquad\qquad A(x)\;\; A(z)A(z)A(z)A(z)\;\; A(y)$$

$$A(x)\;\; A(z)A(z)A(z)A(z)\;\; A(y) \qquad\qquad A(x)\;\; A(z)A(z)A(z)A(z)\;\; A(y) \qquad (7\text{-}45)$$

From (7-45) we glean that for each contraction between two fields at z, we have two subamplitudes. With a little thought you can convince yourself that there are six possible such contractions between the four A fields at the z vertex. Thus, there are a total of 12 such subamplitudes. But since we have a factor of ½ now that we didn't have with the $g^2A^2B^2$ term., the factor in front of this subamplitude integral is 12 X ½ = 6. (We had a factor of two in (7-37).) Thus,

$$-i\Pi_A^{(A)} = 6g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2} \;. \qquad \text{repeat of (7-39)}$$

[2] Introduction to Supersymmetry, Stephen P. Martin, https://www.niu.edu/spmartin/preSUSY2019_Martin.pdf, pg. 12/ .

8 Superfields

This section comprises an introduction to, and overview of, superfields, as treated in Aitchison Chap. 6, plus a supplying of missing steps in certain parts of that chapter.

8.1 Preliminary Concepts

We need to get two ideas under our belts before diving into superfields.

8.1.1 Invariant Subspaces and SUSY

Invariant Subspaces in the Standard Model

If the following is not clear, you may wish to read/review Klauber, Vol. 2, Chap.2, pg. 71, subsection "Full Expressions of Typical Fields in QFT", and Chap. 14, Sects 14.1.2 to 14.1.7, pgs. 419-423.

From (2-110) of the first citation in the prior paragraph, as an example, we express the quantum field for a LC green up quark.

$$\text{LC green up quark}\quad \Psi^L_{ug} = \Psi^L_{12} = \underbrace{\sum_{r,\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left(c_{ur}(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d^\dagger_{ur}(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right)}_{\text{general solution to Dirac equation for up quark, }\psi_u}\begin{bmatrix}1\\0\end{bmatrix}_W\begin{bmatrix}0\\1\\0\end{bmatrix}_S = \psi_u\begin{bmatrix}1\\0\end{bmatrix}_W\begin{bmatrix}0\\1\\0\end{bmatrix}_S \qquad (8\text{-}1)$$

The field has spacetime parts (the $\pm ipx$ exponents and the spinors u_r and v_r), up quark parts (the c_{ur} and $d^\dagger{}_{ur}$ parts), a weak isospin part (the two-component doublet), and a strong force part (the three component color triplet). Somewhat hidden is the electric charge part, for which a charge operator exists whose eigenvalue for (8-1) is +2/3.

Translation and Lorentz transformation operators act on the $e^{\pm ipx}$ parts in 4D spacetime. Spin operators operate on the spinors in 4D spinor space. The weak isospin operators (the Pauli matrices) operate in $SU(2)$ space on the doublet. Strong interaction operators (the Gell-Mann matrices) operate in $SU(3)$ space on the triplet.

A Lorentz transformation will change x^μ, but not spin, nor electric charge, nor up to down, nor green to red or blue. A Gell-Mann operator can only change color, i.e., the component of the triplet. It can't change x^μ, spin, electric charge, or the weak doublet component (up to down). Each space has its own operations associated with it. And none of these operators affect changes in any other space than the one it operates within. (For the purist, we are ignoring complications after symmetry breaking for $U(1)$ electric charge and $SU(2)$ weak isospin, but considering $U(1)$ hypercharge and $SU(2)$ weak isospin spaces.)

The key point is that none of these operators changes things outside of the particular space it operates in. The domain of spacetime is called an invariant subspace of Ψ. The same goes for the domains of each of the $U(1)$, $SU(2)$, and $SU(3)$ spaces. Mathematically, we say the domains of the component operators are mutually orthogonal and taken together, they span the domain of Ψ, which can therefore be expressed as an outer product.

$$\Psi = \Psi_{4D}\otimes\Psi_{U(1)}\otimes\Psi_{SU(2)}\otimes\Psi_{SU(3)}\,, \qquad (8\text{-}2)$$

where a given operator only acts on one space without changing anything in the others.

How SUSY is Different

As we have seen earlier (seven steps in Sect. 6.1, pg. 25), we follow similar steps to develop any QFT. In dong that, we find the symmetries of the Lagrangian under transformations of various fields, or more precisely, under transformations of various subspace fields as in the RHS of (8-2).

As one example, we can consider in $SU(2)$ isospin space of the first Pauli matrix (the first generator of the $SU(2)$ group), what happens to an up quark.

$$\sigma_1\Psi^L_{ug} = \begin{bmatrix} & 1\\ 1 & \end{bmatrix}\psi_u\begin{bmatrix}1\\0\end{bmatrix}_W\begin{bmatrix}0\\1\\0\end{bmatrix}_S = \psi_u\left(\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_W\right)\begin{bmatrix}0\\1\\0\end{bmatrix}_S = \psi_d\begin{bmatrix}0\\1\end{bmatrix}_W\begin{bmatrix}0\\1\\0\end{bmatrix}_S \qquad (8\text{-}3)$$

The operation has changed our "vector" (doublet in "group theory speak") from up to down. So, operations in a given space (like, by analogy, a rotation of a vector in 2D originally aligned along one axis) can change the "vector" (multiplet) in that space. And this changes the character of the subspace multiplet from a particle with a particular property to different such property, in this case up to down.

SUSY is different in that a SUSY operation, as we will see, can change a field, not just inside the SUSY subspace, but outside it, as well. That is, a transformation changing spin type (say from LC to scalar) will also effect a change in position x^μ of the field. The SUSY transformation, though formulated as acting in the SUSY subspace, will actually affect parts of the field outside of that subspace.

Recall our (infinitesimal) SUSY transformation in its 2D space, from (6-29) (pg. 33), also shown in Wholeness Chart 6-2, pg. 26,

$$\delta_\xi \begin{bmatrix} \phi \\ \chi \end{bmatrix} = \begin{bmatrix} & \xi\,\bullet \\ -i\sigma^\mu \bar{\xi} \partial_\mu & \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} + \begin{bmatrix} 0 \\ \xi F \end{bmatrix}. \tag{8-4}$$

This switches spinor fields with scalar fields, and one would expect that is all it does, that it does nothing to other parts of the field, such as color charge, weak isospin charge, and spacetime.

The surprise, as Aitchison and we deduce, turns out to be that (8-4) actually also affects spacetime. It leads to a translation in 4D. It affects a space outside its own subspace. And in fact, since SUSY operations are internal transformations, we find an internal transformation leads to variations in an external space. Details to follow after this Sect. 8.1.

8.1.2 Two Ways for Operators to Act on Fields

Field Operators vs State Operators

Recall from Klauber, Vol. 2, Wholeness Chart 14-1, pg. 423 and associated text, as well as from http://www.quantumfieldtheory.info/Opers_Fields_States.pdf, that we deal with two different spaces in QFT, the space of states, e.g., $|\phi\rangle$, and the space of fields, e.g., ϕ. Each has operations performed on them.

For example, as shown in the link of the prior paragraph, we have a spin operator that acts on fields plus a spin operator that acts on states. They are related, but different.

In SUSY, we don't talk much about the SUSY operation (8-4) on the Hilbert space of fields. We do talk a lot about the SUSY operations Q_1 and Q_2 on the Fock space of states. Like the two kinds of spin operators, they are related, but different in character.

Representing Transformations of Operations on Fields

The Concept

As noted, there are two types of operations, on fields and on states. But, now we want to talk about two ways to express the same operation on just fields - two ways to do one of the types of operations (the field operations) described in the prior subsection. We will find that, for fields, the same operation (variation in the field) can be described mathematically as, either

1. a commutator (or anticommutator) which includes the field, or
2. a differential operator acting on the field.

How the Commutator Description Works

Recall that group operations in a given space are effected by the generators of the algebra of that space. Although we said we would ignore states, we need to bring them in at this point as an intermediary step. So, consider a unitary operator U acting on a state (unprimed) and transforming it to a different state (primed).

$$U|\Psi\rangle = |\Psi'\rangle\,. \tag{8-5}$$

Now consider another operator acting within the same subspace of the state space as U, which we call O_{op}. Then, this operator changes under U via

$$U^{-1} O_{\text{op}} U = O'_{\text{op}}\,. \tag{8-6}$$

U can be expressed in terms of the generators G_i of the algebra of the state subspace as (8-7). (Note I use a minus sign in front of $i\varepsilon_i$, as I believe the result is easier to follow than that of Aitchison, who uses a plus sign in his comparable equation (6.5).)

$$U = e^{-i\varepsilon_1 G_1} e^{-i\varepsilon_2 G_2} e^{-i\varepsilon_3 G_3} \ldots \quad \xrightarrow[\varepsilon_i]{\text{infinitesimal}} \quad U = e^{-i\varepsilon_i G_i} = 1 - i\varepsilon_i G_i\,. \tag{8-7}$$

So, (8-6) becomes

$$\begin{aligned} U O_{\text{op}} U^{-1} &= \left(1 - i\varepsilon_i G_i\right) O_{\text{op}} \left(1 + i\varepsilon_j G_j\right) = O_{\text{op}} - i\varepsilon_i G_i O_{\text{op}} + i O_{\text{op}} \varepsilon_j G_j + \textit{higher order} \\ &= O_{\text{op}} + i\varepsilon_i \left[O_{\text{op}}, G_i\right] = O_{\text{op}} + \delta O_{\text{op}}\,. \end{aligned} \tag{8-8}$$

Hence, the (infinitesimal) variation in the operator O_{op} under the transformation U with generators for U, G_i, is

$$\delta O_{\text{op}} = i\varepsilon_i \left[O_{\text{op}}, G_i \right]. \tag{8-9}$$

The generators "generate" the transformation via a commutator.

How the Derivative Description Works

Consider the case where the operator O_{op} is a function of one or more independent variables, which we label ρ_i, which under the transformation U change(s) by ε_i. That is,

$$O_{\text{op}} = O_{\text{op}}\left(\rho_i\right) \tag{8-10}$$

where, under the infinitesimal ($\varepsilon_i << 1$) transformation,

$$\rho_i \rightarrow \rho_i' = \rho_i + \varepsilon_i . \tag{8-11}$$

We can thus express the change in O_{op} under the transformation in a different way than (8-9), via

$$O_{\text{op}}' = O_{\text{op}} + \varepsilon_i \frac{\partial}{\partial \rho_i} O_{\text{op}} + higher\, order = O_{\text{op}} + \delta O_{\text{op}} . \tag{8-12}$$

So, for infinitesimal transformations

$$\delta O_{\text{op}} = \varepsilon_i \frac{\partial O_{\text{op}}\left(\rho_i\right)}{\partial \rho_i} . \tag{8-13}$$

<u>The Result</u>

So, from (8-9) and (8-13), which are both true,

$$\delta O_{\text{op}} = \varepsilon_i \frac{\partial O_{\text{op}}\left(\rho_i\right)}{\partial \rho_i} = i\varepsilon_i \left[O_{\text{op}}, G_i \right]. \tag{8-14}$$

which is what we claimed at the start. The variation in an operator under a transformation can be expressed as a derivative or as a commutator.

<u>A Particular Operator: Translation in Spacetime</u>

Consider a simple transformation, translation in spacetime,

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \varepsilon^{\mu} \qquad \text{or for covariant components } \; x_{\mu} \rightarrow x_{\mu}' = x_{\mu} + \varepsilon_{\mu} . \tag{8-15}$$

A scalar field $\phi(x^{\mu})$, which in QFT is an operator like O_{op}, doesn't change its value ϕ under translation (as scalars don't, though vectors and spinors do change components). But, it does change its argument x^{μ},

$$\phi\left(x^{\mu}\right) \xrightarrow{\text{translation}} \phi\left(x'^{\mu}\right) = \phi\left(x^{\mu}\right) + \delta\phi\left(x^{\mu}\right). \tag{8-16}$$

There is one generator for each spacetime direction μ, we'll label as G^{μ}, so (8-14) becomes

$$\delta\phi = \varepsilon_{\mu}\left(\frac{\partial}{\partial x_{\mu}} \phi\left(x_{\mu}\right) \right) = i\varepsilon_{\mu}\left[\phi, G^{\mu} \right]. \tag{8-17}$$

Consider (8-17) acting on a state (as fields do), where we use the chain rule in going to the second line and G^{μ} operates on both the state and the field, but ϕ only operates on the state,

$$\begin{aligned} \delta\phi \left| state \right\rangle &= \left(\varepsilon_{\mu} \frac{\partial}{\partial x_{\mu}} \phi \right) \left| state \right\rangle = \left(i\varepsilon_{\mu}\phi G^{\mu} - i\varepsilon_{\mu} G^{\mu}\phi \right) \left| state \right\rangle = i\varepsilon_{\mu}\phi G^{\mu} \left| state \right\rangle - i\varepsilon_{\mu} G^{\mu}\phi \left| state \right\rangle \\ &= i\varepsilon_{\mu}\phi\left(G^{\mu} \left| state \right\rangle \right) - i\varepsilon_{\mu}\left(G^{\mu}\phi \right) \left| state \right\rangle - i\varepsilon_{\mu}\phi\left(G^{\mu} \left| state \right\rangle \right) = -i\varepsilon_{\mu}\left(G^{\mu}\phi \right) \left| state \right\rangle . \end{aligned} \tag{8-18}$$

From the expression after the first equal sign in (8-18) and the last expression therein,

$$\frac{\partial}{\partial x_\mu}\phi = -i\left(G^\mu \phi\right) \qquad \rightarrow \qquad i\frac{\partial}{\partial x_\mu}\phi = G^\mu \phi \qquad \rightarrow \qquad G^\mu = i\frac{\partial}{\partial x_\mu} = \hat{P}^\mu . \tag{8-19}$$

We relabeled G^μ as $\hat{P}^\mu$, since it turns out to be essentially the 4-momenutm operator operating on a field. The caret (hat) on P^μ is used by Aitchison to imply a differential operator, and we will follow that symbolism here.

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ip^\mu x_\mu} \qquad \rightarrow \qquad \hat{P}^\mu \phi = i\frac{\partial}{\partial x_\mu}\phi = p^\mu \phi . \tag{8-20}$$

We started simply by taking G^μ as our generator for spacetime translations of states, without knowing its form as in (8-19). But we showed, via (8-19) and (8-20), that it is actually the 4-momentum operator on fields.

(Note Aitchison sticks a minus sign in his (6.10) [89], in order to get the RHS of (8-20), whereas we used a minus sign in (8-7), which is essentially arbitrary, since ε_i can be positive or negative. IMO, it comes out with less head scratching the way we have done it here.)

Summary of Ways to Represent Change in an Operator

Bottom line: For an operator (O_{op} in general, ϕ in our special case) that is a function of a variable (ρ_i) in general, x_μ in our special case), where a transformation arises from a change in that variable, the change in the operator can be expressed in two ways, via a commutator and via a derivative. We repeat (8-14) for the general case below. See (8-17) (with the RHS of (8-19)) for our special case.

$$\delta O_{\text{op}} = \varepsilon_i \frac{\partial O_{\text{op}}(\rho_i)}{\partial \rho_i} = i\varepsilon_i \left[O_{\text{op}}, G_i\right] \qquad \delta\rho_i = \varepsilon_i \qquad \hat{G}_i = i\frac{\partial}{\partial \rho_i} \qquad \begin{array}{l} G_i = \text{ generators of the algebra} \\ \hat{G}_i = \text{ differential operator form of generators} \end{array} . \tag{8-21}$$

8.2 Deducing Aitchison Eq (6.28)

The operation of spacetime translation (via 4D translation operator P), plus operation of the SUSY superspace spinor (two components) operator Q on LC states, plus the operation of the SUSY superspace spinor (two components) operator $Q^\dagger$ (other notation Q^* or $\overline{Q}$) on RC states is

$$U\left(x,\theta,\theta^*\right) = e^{ix\cdot P} e^{i\theta\cdot Q} e^{i\bar{\theta}\cdot\bar{Q}} \qquad \text{Aitchison (6.12) [90].} \tag{8-22}$$

The overbar indicates RC operation (on the top two components of a state in the Weyl rep); no overbar indicates LC operation (on the bottom two components of a state in the Weyl rep);

Note the inner product of two LC spinors in the 3rd row from the bottom, 2nd column of Wholeness Chart 5-2, pg. 21, and from that,

$$\begin{aligned} \theta\cdot Q &\equiv \theta^a Q_a = \theta_2 Q_1 - \theta_1 Q_2 = \begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\begin{pmatrix}-Q_2 \\ Q_1\end{pmatrix} = \begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{pmatrix}Q_1 \\ Q_2\end{pmatrix}\right) \\ &= \theta^T\left(-i\sigma_2\right)Q \qquad \text{where} \quad \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = -i\sigma_2 \qquad \text{Aitchison } (6.13)\ [90] \end{aligned} \tag{8-23}$$

Note the inner product of two RC spinors in the 3rd row from the bottom and the very bottom row, 3rd column of Wholeness Chart 5-2, pg. 21, and from that,

$$\begin{aligned} \bar{\theta}\cdot\bar{Q} &\equiv \theta^*_{\dot{a}} Q^{*\dot{a}} = -\theta^{*\dot{2}} Q^{*\dot{1}} + \theta^{*\dot{1}} Q^{*\dot{2}} = \begin{pmatrix}\theta^{*\dot{1}} & \theta^{*\dot{2}}\end{pmatrix}\begin{pmatrix}Q^{*\dot{2}} \\ -Q^{*\dot{1}}\end{pmatrix} = \begin{pmatrix}\theta^{*\dot{1}} & \theta^{*\dot{2}}\end{pmatrix}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{pmatrix}Q^{*\dot{1}} \\ Q^{*\dot{2}}\end{pmatrix}\right) \\ &= \theta^{*T}\left(i\sigma_2\right)Q^* = \theta^\dagger\left(i\sigma_2\right)Q^{\dagger T} \qquad \text{where} \quad \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = i\sigma_2 . \qquad \text{Aitchison } (6.14)\ [90] \end{aligned} \tag{8-24}$$

So, the action of (8-22) on a field Φ (which itself is an operator on states), where the initial field had $x=\theta=\theta^*=0$, is

$$U\left(x,\theta,\theta^*\right)\Phi(0)U^{-1}\left(x,\theta,\theta^*\right) = \Phi\left(x,\theta,\theta^*\right) \qquad \text{Aitchison (6.12) [90].} \tag{8-25}$$

Now, we examine the action of two successive operations UU.

$$U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)=e^{ia\cdot P}e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{ix\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} \qquad \text{Aitchison (6.17) [91]} \quad (8\text{-}26)$$

From **Error! Reference source not found.**, the P operators commute with Q and $Q^\dagger$, so

$$U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)=e^{ia\cdot P}e^{ix\cdot P}\underbrace{e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}}_{\boxed{1}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}}. \tag{8-27}$$

To evaluate the last four factors, one needs the Baker-Campbell-Hausdorff (BCH) relation

$$e^Ae^B=e^{A+B+\frac{1}{2}[A,B]+\frac{1}{6}[[A,B],B]+\dots}. \tag{8-28}$$

Aitchison, in (6.20) to (6.23) shows the third and fourth factors in (8-27) to be

$$\boxed{1}=e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}=e^{\frac{i}{2}\left(i\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P}e^{i\xi\cdot Q+i\bar{\xi}\cdot\bar{Q}}, \tag{8-29}$$

turning (8-27) into

$$\begin{aligned}U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)&=e^{ia\cdot P}e^{ix\cdot P}\underbrace{e^{\frac{i}{2}\left(i\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P}e^{i\xi\cdot Q+i\bar{\xi}\cdot\bar{Q}}}_{\boxed{1}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}}\\&=e^{ia\cdot P}e^{ix\cdot P}e^{\frac{i}{2}\left(i\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P}\underbrace{e^{i\xi\cdot Q+i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}}_{\boxed{2}}e^{i\bar{\theta}\cdot\bar{Q}}.\end{aligned} \tag{8-30}$$

$$\boxed{2}=e^{i\xi\cdot Q+i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}=e^{i\left(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q\right)+i^2\frac{1}{2}\left[\xi\cdot Q+\bar{\xi}\cdot\bar{Q},\theta\cdot Q\right]+\dots}=e^{i\left(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q\right)-\frac{1}{2}\left[\xi\cdot Q,\theta\cdot Q\right]-\frac{1}{2}\left[\bar{\xi}\cdot\bar{Q},\theta\cdot Q\right]+\dots} \tag{8-31}$$

Q commutes with itself, so $\left[\xi\cdot Q,\theta\cdot Q\right]=0$. Aitchison (reproduced in the appendix herein) evaluated the last commutator shown in (8-31) to deduce his (6.21), part of our (8-29), where we take ξ there equal to θ here.

$$\left[\theta\cdot Q,\bar{\xi}\cdot\bar{Q}\right]=\theta^a\xi^{*b}\left(\sigma^\mu\right)_{ab}P_\mu \qquad \text{Aitchison (6.21) [91].} \quad (8\text{-}32)$$

With (8-32), (8-31) becomes

$$\boxed{2}=e^{i\left(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q\right)-\frac{i}{2}\left(i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P+\dots}. \tag{8-33}$$

Since P_μ commutes with Q and $Q^\dagger$, we can move it to the front of $\boxed{2}$ in (8-33) and thus, in (8-30). This makes (8-30) into

$$\begin{aligned}U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)&=e^{ia\cdot P}e^{ix\cdot P}e^{\frac{i}{2}\left(i\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}\underbrace{e^{-\frac{i}{2}\left(i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P+i\left(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q\right)+\dots}}_{\boxed{2}}e^{i\bar{\theta}\cdot\bar{Q}}\\&\approx e^{ia\cdot P}e^{ix\cdot P}e^{\frac{i}{2}\left(i\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}e^{-\frac{i}{2}\left(i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P}\underbrace{e^{i\left(\left(\xi+\theta\right)\cdot Q+\bar{\xi}\cdot\bar{Q}\right)}}_{\boxed{3}}e^{i\bar{\theta}\cdot\bar{Q}}.\end{aligned} \tag{8-34}$$

For $\boxed{3}$, consider the BCH relation

$$e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}=e^{i\left(\left(\xi+\theta\right)\cdot Q+\bar{\xi}\cdot\bar{Q}\right)-\frac{1}{2}\left[\left(\left(\xi+\theta\right)\cdot Q,\bar{\xi}\cdot\bar{Q}\right)\right]} \;\rightarrow\; e^{i\left(\left(\xi+\theta\right)\cdot Q+\bar{\xi}\cdot\bar{Q}\right)}=e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}e^{\frac{1}{2}\left[\left(\left(\xi+\theta\right)\cdot Q,\bar{\xi}\cdot\bar{Q}\right)\right]}=\boxed{3}\;.(8\text{-}35)$$

The commutator in (8-35), similar to (8-32), is

$$\left[\left(\xi+\theta\right)\cdot Q,\bar{\xi}\cdot\bar{Q}\right]=\left(\xi+\theta\right)^a\left(\sigma^\mu\right)_{ab}\xi^{*b}P_\mu, \tag{8-36}$$

so,

$$\boxed{3}=e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}e^{\frac{1}{2}\left(\xi+\theta\right)^a\xi^{*b}\left(\sigma^\mu\right)_{ab}P_\mu}=e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}e^{\frac{1}{2}\xi^a\xi^{*b}\left(\sigma^\mu\right)_{ab}P_\mu}e^{\frac{1}{2}\theta^a\xi^{*b}\left(\sigma^\mu\right)_{ab}P_\mu}. \tag{8-37}$$

With (8-37) in (8-34), we have, where, as always, we commute P_μ with Q and $Q^\dagger$ (essentially $\bar{Q}$),

$$U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)$$

$$= e^{ia\cdot P}e^{ix\cdot P_\mu}\underbrace{e^{-\frac{1}{2}\left(\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}}_{\text{cancels}}\underbrace{e^{\frac{1}{2}\left(\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}}_{\text{exponents add}}\overbrace{e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}\underbrace{e^{\frac{1}{2}\xi^a\left(\sigma^\mu\right)_{ab}\xi^{*b}P_\mu}}_{\text{cancels}}\underbrace{e^{\frac{1}{2}\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}P_\mu}}_{\text{exponents add}}}^{\boxed{3}}e^{i\bar{\theta}\cdot\bar{Q}} \tag{8-38}$$

$$= e^{ia\cdot P}e^{ix\cdot P_\mu}e^{\left(\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\bar{\xi}\cdot\bar{Q}\right)}e^{i\bar{\theta}\cdot\bar{Q}},$$

or, finally,

$$U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right) = e^{ia\cdot P}e^{ix\cdot P_\mu}e^{\left(\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\left(\bar{\xi}+\bar{\theta}\right)\cdot\bar{Q}\right)}$$

$$= e^{ia\cdot P}e^{ix\cdot P_\mu}\underbrace{e^{i\left(-i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}\right)\cdot P_\mu}e^{i\left(\left(\xi+\theta\right)\cdot Q\right)}e^{i\left(\left(\bar{\xi}+\bar{\theta}\right)\cdot\bar{Q}\right)}}_{\text{Aitchison (6.28) [92]}}. \tag{8-39}$$

Thus, the action of (8-39) on a scalar field is

$$U\left(a,\xi,\xi^*\right)U\left(x,\theta,\theta^*\right)\Phi\left(0,0,0\right)U^{-1}\left(x,\theta,\theta^*\right)U^{-1}\left(a,\xi,\xi^*\right)$$

$$= U\left(a,\xi,\xi^*\right)\Phi\left(x,\theta,\theta^*\right)U^{-1}\left(a,\xi,\xi^*\right) \qquad \text{Aitchison (6.30) [93]} \tag{8-40}$$

$$= \Phi\left(x^\mu + a^\mu - i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b},\theta+\xi,\theta^*+\xi^*\right).$$

This means the action of U done twice successively results in the transformations of the arguments of the field Φ as on the right side of (8-41).

$$0 \xrightarrow{U\left(x^\mu,\theta,\theta^*\right)} x^\mu \xrightarrow{U\left(x^\mu,\xi,\xi^*\right)} \underbrace{x^\mu + a^\mu}_{\text{expected}}\;\underbrace{-i\theta^a\left(\sigma^\mu\right)_{ab}\xi^{*b}}_{\text{unexpected}}$$

$$0 \xrightarrow{U\left(x^\mu,\theta,\theta^*\right)} \theta \xrightarrow{U\left(x^\mu,\xi,\xi^*\right)} \theta+\xi \qquad \left(\text{expected}\right) \qquad \text{Aitchison (6.29) [92]} \tag{8-41}$$

$$0 \xrightarrow{U\left(x^\mu,\theta,\theta^*\right)} \theta^* \xrightarrow{U\left(x^\mu,\xi,\xi^*\right)} \theta^*+\xi^* \qquad \left(\text{expected}\right)$$

The unexpected part of (8-41) arose from the parts of U containing Q (and its complex conjugate), not from the spacetime part containing P_μ.

<u>Bottom line:</u> A purely SUSY transformation (two transformations, actually) on a field results in a transformation (translation) in spacetime.

<u>Interpretation</u>

We might think of expressing, in manner similar to (8-2), the SUSY part of the field as an outer product, as in (8-42). But it would be incorrect to use the ⊗ symbol, because the SUSY and spacetime parts do not comprise invariant subspaces. They actually mix, as we showed above in (8-41).

$$\Psi = \Psi_{4D} \otimes \Psi_{U(1)} \otimes \Psi_{SU(2)} \otimes \Psi_{SU(3)} \overset{\downarrow\text{incorrect}}{\otimes} \Psi_{SUSY} \tag{8-42}$$

SUSY actually changes 4D

We might have suspected this, given that commutation/anticommutation relations for each of the four components of a standard model field only involve field operators within its own particular invariant subspace. For example, the $SU(2)$ commutators only involve the $SU(2)$ operators, the Pauli matrices, i.e.,

$$\left[\sigma_i,\sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k. \tag{8-43}$$

For SUSY, on the other hand, we have (6-72), repeated below.

$$\left[Q_a,Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab}P_\mu. \tag{8-44}$$

It is (8-44) that results in (8-41), shown symbolically in (8-42).

Remember that the Ψ_{SUSY} part of the field is a doublet comprising a particle's field as one component plus its spartner field as the other component. We have been using Φ for the field in what we have been doing so far, rather than Ψ. Φ represents a scalar (with a SUSY sfermion spartner obtained via the SUSY transformation Q_a), whereas Ψ is used herein and elsewhere as a more general field (scalar, fermion, gauge boson).

We can think of the field Ψ in (8-42) (or our Φ in what we have been doing) as a vector in a large multidimensional Hilbert space, where some axes are spacetime x^μ, some are axes in $SU(2)$ space, some in $SU(3)$ space, etc. The direction the field points, figuratively, in this multidimensional space pins down what field we are talking about. The quantum numbers (eigenvalues of operators for hypercharge, isospin, etc.) are each on a separate axis, and whatever they are determines what the field is. Different quantum numbers mean different alignments of the Ψ "vector" in this higher dimensional space.

This Ψ vector can be visualized as rotated in the Hilbert space when operators act on it. An $SU(2)$ operation (via Pauli matrices), for example, can change a LC down quark into a LC up quark, which can be visualized as a vector pointing along one $SU(2)$ axis. The down quark could also, however, be "rotated" into a state which is a superposition of the up and down states. It would then be pointed, figuratively, at some angle to both the up axis and the down axis in $SU(2)$ space.

A SUSY operation on Ψ would "rotate" it between a field and its spartner field, either completely, or in the case of superposition, to some state which is mixture of the two. But, whereas an $SU(2)$ transformation doesn't change the spacetime argument x^μ, the SUSY transformation does, via (8-41). So, visually and figuratively, the SUSY transformation rotates the vector Ψ out of the subspace one would otherwise consider it to be confined to.

In our case, we have used $\Psi = \Phi$, and in (8-25), we turn a scalar $\Phi(x^\mu = 0)$ with no dependence on x^μ, θ, or θ^* into a field with such dependence $\Phi(x^\mu, \theta, \theta^*)$. Thus, in our Hilbert space, it has spacetime positions along four axes of x^0, x^1 x^2, and x^3; and in SUSY space it has values along four axes of θ_1, θ_2, $\theta_1{}^*$, $\theta_2{}^*$. It also has values along all the $U(1)$, $SU(2)$, and $SU(3)$ space axes, though those are usually suppressed when we are developing SUSY theory and use the symbol $\Phi(x^\mu, \theta, \theta^*)$.

Thus, unlike our usual scalar fields in QFT, operations in SUSY space don't affect just the values of θ_1, θ_2, $\theta_1{}^*$, $\theta_2{}^*$. They also affect x^μ. In effect, the vector Φ is "rotated" partially out of the SUSY part of the space. This "rotation" changes (adds or subtracts values from) the x^μ axes. $SU(2)$ and other operations do not rotate Φ out of $SU(2)$ (or whatever other) space. $SU(2)$ "rotations" do not change values along the $SU(3)$ axis, or the x^μ axes. That SUSY transformations do this is a unique feature of SUSY theory.

Definitions. $\Phi(x^\mu, \theta, \theta^*)$ is called a superfield, as it has components of both a scalar and its sfermion partner *plus* it is a function of eight variables x^0, x^1 x^2, x^3, θ_1, θ_2, $\theta_1{}^*$, and $\theta_2{}^*$, spanning an eight dimensional space. That space is called superspace.

Note that the four dimensions with numeric values θ_1, θ_2, $\theta_1{}^*$, and $\theta_2{}^*$ comprise a space of Grassmann variables, as those values obey anti-commutation relations. The x^μ, on the other hand, are normal, commuting numbers.

In effect, the arena of spacetime we are so familiar with has been merged with another space and enlarged. This is why SUSY is often described as having an additional four dimensions of spacetime. There are two ways to summarize the theory.

1. As a system of particles, each with a spartner ($N = 1$ SUSY) having all the same properties except spin, or
2. as an eight dimensional theory, four being spacetime, with superfields that comprise both bosons and fermions and live in all eight dimensions.

Note that the term "superspace" to define all eight dimensions is standard terminology. Herein, we often use the term "SUSY space" to refer specifically to the space of the θ and θ^*, i.e., the arena of just the SUSY operators Q_a and $Q_a{}^\dagger$. That is different from "superspace".

8.3 SUSY Operators as Derivatives

8.3.1 The Objective and Procedure

This section summarizes Sect. 6.2, pgs. 93-95, in Aitchison, plus elaborates (hopefully to clarify) on parts of that section I found confusing. In essence, for the SUSY operators Q_a and $Q^\dagger{}_a$ used in commutators, we find their derivative forms, symbolized by $\hat{Q}_a$ and $\hat{Q}^\dagger{}_a$.

The procedure entails transforming a scalar $\Phi(x^\mu = 0)$ originally at the spacetime origin that has no SUSY components via $U(x^\mu, \theta, \theta^*)$ of (8-41) to get a scalar $\Phi(x^\mu, \theta, \theta^*)$. We then work with that latter scalar as a more general form of Φ, i.e., not a particular one that has $x^\mu = \theta = \theta^* = 0$. We then operate on $\Phi(x^\mu, \theta, \theta^*)$ via $U(a^\mu, \xi, \xi^*)$ of (8-41), and examine its variation $\delta\Phi(x^\mu, \theta, \theta^*)$ under that second transformation† .

8.3.2 Finding Differential Forms for SUSY Operators Q_a and $Q^\dagger{}_a$

The change in $\Phi(x^\mu, \theta, \theta^*)$, via the first relation in (8-17), is

$$\delta\Phi(0,0,0) = \delta x^\mu \frac{\partial\Phi}{\partial x^\mu} + \delta\theta^a \frac{\partial\Phi}{\partial\theta^a} + \delta\theta_a^* \frac{\partial\Phi}{\partial\theta_a^*} . \qquad (8\text{-}45)$$

With (8-41), this becomes

$$\delta\Phi(x^\mu,\theta,\theta^*) = \left(a^\mu - i\theta^a (\sigma^\mu)_{ab}\, \xi^{*b}\right)\frac{\partial\Phi}{\partial x^\mu} + \xi^a \frac{\partial\Phi}{\partial\theta^a} + \xi_a^* \frac{\partial\Phi}{\partial\theta_a^*} . \qquad (8\text{-}46)$$

Recall the a^μ part of (8-46) came from the transformation of x^μ, but the other part inside the parentheses arose from the transformations on θ^a and $\theta_a{}^*$. We will focus on the SUSY transformations, employing θ^a and $\theta_a{}^*$, not the spacetime transformation, so we now consider that case, i.e., the case with $a^\mu = 0$ in (8-46). For that, (8-46) becomes

$$\delta\Phi = -i\theta^a (\sigma^\mu)_{ab}\, \xi^{*b} \frac{\partial\Phi}{\partial x^\mu} + \xi^a \frac{\partial\Phi}{\partial\theta^a} + \xi_a^* \frac{\partial\Phi}{\partial\theta_a^*} \qquad \text{Aitchison (6.31) [93].} \qquad (8\text{-}47)$$
$$\left(\text{tranformation only on } \theta^a \text{ and } \theta_a^*, \text{ not } x^\mu\right)$$

Consider (8-21), where here $O_{\text{op}} = \Phi$, $G_{1,2} = Q_a$, $G_{3,4} = Q^\dagger{}_a$, and $\varepsilon_\iota = \xi^a$, $\xi_a{}^*$. Then, comparing to (8-47), one notes there is one term in ξ^a and two terms in $\xi_a{}^*$. The former is a relatively easy deduction,

$$\hat{Q}_a = i\frac{\partial}{\partial\theta^a} . \qquad (8\text{-}48)$$

For the latter we have (note Aitchison is a bit confusing on this, as his (6.38) and (6.39) should be combined)

$$\xi_a^* \hat{Q}^{\dagger a}\Phi = \underbrace{-\theta^a (\sigma^\mu)_{ab}\, \xi^{*b} \frac{\partial\Phi}{\partial x^\mu}}_{\boxed{B}} + \underbrace{\xi_a^* \frac{\partial\Phi}{\partial\theta_a^*}}_{\boxed{A}} \qquad (8\text{-}49)$$

$$\boxed{A} \text{ part of yields } {}_A\hat{Q}^{\dagger a} = i\frac{\partial}{\partial\theta_a^*} \qquad \boxed{B} \text{ part of yields } -i\xi_a^*{}_B\hat{Q}^{\dagger a}\Phi = -i\theta^a (\sigma^\mu)_{ab}\, \xi^{*b} \frac{\partial\Phi}{\partial x^\mu} \qquad (8\text{-}50)$$

Aitchison then uses the properties of Grassman variables [pg. 93-93] to convert $\boxed{B}$ and thus, convert (8-50) to

$${}_A\hat{Q}^{\dagger a} +_B \hat{Q}^{\dagger a} = \hat{Q}^{\dagger a} \rightarrow \quad \hat{Q}_a^\dagger = -i\frac{\partial}{\partial\theta^{a*}} + \theta^b (\sigma^\mu)_{ba} \frac{\partial}{\partial x^\mu} \qquad \text{Aitchison (6.41) [94].} \qquad (8\text{-}51)$$

(8-48) and (8-51) are the expressions we were seeking: the SUSY operators expressed as derivatives.

Homework Problem 8-1. Show that this form of the SUSY operators, (8-48) and (8-51), satisfies the anti-commutation and commutation relations of the SUSY algebra (6-9).

8.4 Chiral Superfields and Their Component Fields

This section summarizes Sect. 6.3 of Aitchison, pgs. 95-97.

We can expand the superfield Φ in powers of θ and θ^*, where we note that any products of form $\theta_{\underline{a}}\theta_{\underline{a}}$ (underline = no sum) or $\theta^*{}_{\underline{a}}\theta^*{}_{\underline{a}}$ equal zero, for Grassmann variables. However, we can simplify, for the time being, by consider $\theta^* = 0$, so we are only dealing with a LC superfield (having only χ fermions and no ξ fermions).

$$\text{LC superfield}\qquad \begin{aligned}\Phi(x,\theta,0) &= \phi(x,0,0) + \theta\cdot\frac{\partial}{\partial\theta}\phi(x,0,0) + \tfrac{1}{2}\theta\cdot\theta\frac{\partial^2}{\partial\theta^2}\phi(x,0,0) \\ &= \phi(x,0,0) + \theta^a\frac{\partial}{\partial\theta^a}\phi(x,0,0) + \tfrac{1}{2}\theta^a\theta_a\frac{\partial^2}{\partial\theta_b\partial\theta^b}\phi(x,0,0)\end{aligned} \tag{8-52}$$

Note that (8-52) is actually exact, and not a lower order approximation, since the higher order terms are all zero.

Aitchison then claims (and later proves in Sect. 6.3) that in (8-52),

$$\theta\cdot\frac{\partial}{\partial\theta}\phi(x,0,0) = \theta\cdot\chi(x,0,0) \quad \text{and} \quad \tfrac{1}{2}\theta\cdot\theta\frac{\partial^2}{\partial\theta^2}\phi(x,0,0) = \tfrac{1}{2}\theta\cdot\theta F(x,0,0), \tag{8-53}$$

so that (8-52) becomes

$$\Phi(x,\theta,0) = \phi(x) + \theta\cdot\chi(x) + \tfrac{1}{2}\theta\cdot\theta F(x) \qquad \text{Aitchison (6.43) [95],} \tag{8-54}$$

where the fields ϕ, χ, and F transform correctly, i.e., as in (6-26) to (6-28).

Bottom line: (8-54) represents a LC superfield containing the components fields ϕ, χ, and F, all of which transform correctly under the SUSY transformations. The chiral superfield is said to provide a linear representation of the SUSY algebra.

Note that Φ comprises a scalar, its SUSY fermion partner, and the auxiliary field, and in addition, lives in a six dimensional space of four x^μ and two θ^a. If we had done a similar thing by expanding in θ and θ^*, we would have $\Phi(x, \theta, \theta^*)$ in terms of a scalar ϕ, an LC fermion χ, an RC fermion ψ, and the auxiliary field F. That superfield would live in an eight dimensional space of four x^μ, two θ^a, and two θ^{*a}.

Note further that even though (8-52) is exact, in the derivation showing it equals (8-54), infinitesimal transformations were employed. So, that latter relation is good only in the small parameter limit.

8.5 Products of Chiral Superfields

This section summarizes Sect. 6.4, pgs 97-100 in Aitchison.

Consider, instead of a single flavor field, a single spartner, and a single F field in (8-54), we consider all flavors, each having a different subscript i. Then

$$\Phi_i(x,\theta,0) = \phi_i(x) + \theta\cdot\chi_i(x) + \tfrac{1}{2}\theta\cdot\theta F_i(x) \qquad \text{Aitchison (6.53) [97].} \tag{8-55}$$

It turns out we can deduce valuable expressions from products of two or three superfields of form (8-55). For example, from

$$\Phi_i\Phi_j = \text{ terms with pairs of fields multiplied .} \tag{8-56}$$

As Aitchison shows, if we examine only the terms that have an F_i factor in them, they have a useful form, as follows, where the bar F means only terms resulting from (8-56) that had an F_i factor.

$$W_{quad} = \tfrac{1}{2}M_{ij}\Phi_i\Phi_j\big|_F = M_{ij}\phi_i F_j - \tfrac{1}{2}M_{ij}\chi\cdot\chi \qquad M_{ij} \text{ symmetric in } i,j. \qquad \text{Aitchison (6.62) \& (6.63) [98].} \tag{8-57}$$

Similarly, for a triple product of superfields

$$\Phi_i\Phi_j\Phi_k, \tag{8-58}$$

one finds

$$W_{cubic} = \frac{1}{6}y_{ijk}\,\Phi_i\Phi_j\Phi_k\big|_F = \frac{1}{2}y_{ijk}\phi_i\phi_j F_k - \frac{1}{2}y_{ijk}\chi_i\cdot\chi_j\phi_k \qquad y_{ijk} \text{ symmetric in } i,j,k \qquad \text{Aitchison (6.68) \& (6.69) [99].} \tag{8-59}$$

Consider

$$\begin{aligned} W = W_{quad} + W_{cubic} &= \tfrac{1}{2}M_{ij}\Phi_i\Phi_j\big|_F + \frac{1}{6}y_{ijk}\,\Phi_i\Phi_j\Phi_k\big|_F \qquad (\leftarrow \text{in terms of superfield } \Phi) \\ &= M_{ij}\phi_i F_j - \tfrac{1}{2}M_{ij}\chi\cdot\chi + \frac{1}{2}y_{ijk}\phi_i\phi_j F_k - \frac{1}{2}y_{ijk}\chi_i\cdot\chi_j\phi_k . \end{aligned} \tag{8-60}$$

From (7-7) (repeated below) for the Wess-Zumino model

$$\mathcal{L}_{\text{int WZ}} = W_i\left(\phi,\phi^\dagger\right)F_i - \tfrac{1}{2}W_{ij}\left(\phi,\phi^\dagger\right)\chi_i \cdot \chi_j + h.c.\,, \tag{8-61}$$

which along with (7-2), (7-3), and (7-4) (repeated below),

$$W_i = \frac{\partial W}{\partial \phi_i} = M_{ij}\phi_j + \tfrac{1}{2}y_{ijk}\phi_j\phi_k \qquad\qquad W_{ij} = \frac{\partial^2 W}{\partial\phi\partial\phi_j} = M_{ij} + y_{ijk}\phi_k\,, \tag{8-62}$$

$$W = \tfrac{1}{2}M_{ij}\phi_i\phi_j + \tfrac{1}{6}y_{ijk}\phi_i\phi_j\phi_k \quad (\leftarrow \text{this } W \text{ for W-Z model}) \tag{8-63}$$

gives us

$$\mathcal{L}_{\text{int WZ}} = M_{ij}\phi_j F_i + \tfrac{1}{2}y_{ijk}\phi_j\phi_k F_i - \tfrac{1}{2}M_{ij}\chi_j \cdot \chi_j - \tfrac{1}{2}y_{ijk}\phi_k\chi_i \cdot \chi_j + h.c. \quad (\leftarrow \text{W-Z model}), \tag{8-64}$$

which equals the second row of (8-60). Considering the potential in the Lagrangian is negative, we find the Wess-Zumino superpotential is $-W$, although the minus sign is typically ignored

$$\text{Wess-Zumino superpotential} = W\,. \tag{8-65}$$

All the interactions for the W-Z model are included in the superpotential W.

Note that the W of this section in the first row of (8-60) has the same form in Φ, as the W used in the W-Z model of (8-63) has in ϕ. That is why we used the same symbol for both in (8-60) and (8-63).

8.6 Other Forms of Chiral Superfield

This section summarizes Sect 6.5 in Aitchison, pgs. 100-105). Note we are dealing here with a general superfield, i.e., it contains both LC and RC fields, not simply LC as we did earlier.

8.6.1 Different Forms of the Superfield Transformation

Recall the SUSY unitary operator (8-22) (repeated below) we started our investigation into superspace with, where the reason for the subscript I will become apparent,

$$U_I\left(x,\theta,\theta^*\right) = e^{ix\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}}\,. \tag{8-66}$$

This yielded a particular form of superfield (see (8-40))

$$\begin{gathered}
U_I\left(x,\theta,\theta^*\right)\Phi(x,0,0)U_I^{-1}\left(x,\theta,\theta^*\right) = \Phi_I\left(x,\theta,\theta^*\right) \\
U_I\left(x,\theta,\theta^*\right)\Phi\left(x,\theta,\theta^*\right)U_I^{-1}\left(x,\theta,\theta^*\right) = \Phi_I\left(x^\mu + a^\mu - i\theta\sigma^\mu\xi^*,\ \theta+\xi,\ \theta^* + \xi^*\right).
\end{gathered} \tag{8-67}$$

But we could, instead of (8-66), have started with

$$U_{II}\left(x,\theta,\theta^*\right) = e^{ix\cdot P}e^{i\bar{\theta}\cdot\bar{Q}}e^{i\theta\cdot Q}\,, \tag{8-68}$$

or

$$U_{real}\left(x,\theta,\theta^*\right) = e^{ix\cdot P}e^{i\left(\theta\cdot Q + \bar{\theta}\cdot\bar{Q}\right)}\,. \tag{8-69}$$

If one goes through the math, one finds, for (8-68),

$$\Phi_{II}\left(x^\mu + a^\mu + i\xi\sigma^\mu\theta^*,\ \theta+\xi,\ \theta^* + \xi^*\right) \tag{8-70}$$

and then, for (8-69),

$$\Phi_{real}\left(x^\mu + a^\mu - \tfrac{1}{2}i\xi\bar{\sigma}^\mu\theta - \tfrac{1}{2}i\theta^\dagger\bar{\sigma}^\mu\xi,\ \theta+\xi,\ \theta^* + \xi^*\right). \tag{8-71}$$

There are three types of superfield, distinguished by the change in the spacetime coordinate, which is different with each different form for U. It turns out that if $\Phi(x,0,0)$ is real, then $\Phi_{real}(x,\theta,\theta^*)$ is also real. In that case, however, Φ_I and Φ_{II} are not real. Hence, the subscript *real* on Φ_{real}.

Each of the three different forms of superfield can be expanded in terms of θ and θ^*, just as we did earlier for Φ_I (symbolized by Φ when we did it). When one does that one gets different terms in ϕ, χ, and F, and in fact with other fields beyond them (such as ψ, for one, since we are no longer restricting ourselves to LC fermions).

However, in each case there would be *irreducible* sets of fields, meaning that the fields in a set would only transform among themselves, and not mix with other fields outside that particular set. For example, when we worked with Φ_I, with $\theta^* = 0$, we found ϕ, χ, and F transformed among themselves, but not with ψ.

8.6.2 The Free Lagrangian

In (8-60) we found the LC interaction terms in the W-Z Lagrangian by multiplying the superfield Φ_I by itself once and twice, where there $\theta^* = 0$. It turns out that one can find the free Lagrangian terms in a similar way (see Aitchison Sect. 6.5), pgs 104-105 for details) by using

$$\left(\Phi^L_{real}\left(x,\theta,\theta^*\right)\right)^\dagger \Phi^L_{real}\left(x,\theta,\theta^*\right). \tag{8-72}$$

Just as we had terms before from the F_i field, and used them to find the interaction terms (see (8-60)), we get a similar result here, but instead of F, such terms are deemed D-component terms. Doing this, one finds, where the D implies picking out the D-component terms in the product,

$$\mathcal{L}_{free,LC} = 4\Phi^{L\,\dagger}_{real}\Phi^L_{real}\Big|_D = \partial^\mu\phi^\dagger\partial_\mu\phi + i\overline{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^\dagger F. \tag{8-73}$$

As Aitchison shows, this can also be found via

$$\mathcal{L}_{free,LC} = \int d^4\theta\,\Phi^{L\,\dagger}_{real}\Phi^L_{real}. \tag{8-74}$$

8.7 Appendix. Reproduction of Derivation of Aitchison (6.21)

$$\begin{aligned}
[\xi\cdot Q, \bar{\xi}\cdot\bar{Q}] &= [\xi^1 Q_1 + \xi^2 Q_2, \xi_1^* Q_2^\dagger - \xi_2^* Q_1^\dagger] \\
&= [\xi^1 Q_1 + \xi^2 Q_2, -\xi^{2*} Q_2^\dagger - \xi^{1*} Q_1^\dagger] \\
&= [\xi^a Q_a, -\xi^{b*} Q_b^\dagger] \\
&= -\xi^a Q_a \xi^{b*} Q_b^\dagger + \xi^{b*} Q_b^\dagger \xi^a Q_a \\
&= \xi^a \xi^{b*}(Q_a Q_b^\dagger + Q_b^\dagger Q_a) \\
&= \xi^a \xi^{b*}(\sigma^\mu)_{ab} P_\mu \qquad \text{Aitchison (6.21) [91]}
\end{aligned}$$

8.8 Solution to Problem

Homework Problem 8-1. Show that the forms of the SUSY operators (8-48) and (8-51) satisfy the anti-commutation and commutation relations of the SUSY algebra (6-9).

Ans.

$$\begin{aligned}
&\left[Q_a, Q_b\right]_+ = 0 \\
&\left[Q_a, P_\mu\right] = \left[Q_a^\dagger, P_\mu\right] = 0 \qquad \text{repeat of (6-9)} \\
&\left[Q_a, Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab} P_\mu
\end{aligned}$$

$$\hat{Q}_a = i\frac{\partial}{\partial\theta^a} \qquad \text{repeat of (8-48)}$$

$${}_A\hat{Q}^{\dagger a} + {}_B\hat{Q}^{\dagger a} = \hat{Q}^{\dagger a} \;\rightarrow\; \hat{Q}_a^\dagger = -i\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\mu\right)_{ba}\frac{\partial}{\partial x^\mu} \qquad \text{repeat of (8-51)}$$

From the first row of (6-9) and (8-48), where θ^a and θ^b are spinor quantities, and anti-commute, so derivatives with respect to them anti-commute, as well.

$$\left[\hat{Q}_a, \hat{Q}_b\right]_+ = i\frac{\partial}{\partial\theta^a}i\frac{\partial}{\partial\theta^b} + i\frac{\partial}{\partial\theta^b}i\frac{\partial}{\partial\theta^a} = -\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^b} - \frac{\partial}{\partial\theta^b}\frac{\partial}{\partial\theta^a} = -\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^b} + \frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^b} = 0. \tag{8-75}$$

From the second row of (6-9) and (8-48), where we note that P^μ equals (8-19) and that spinors (and derivatives with respect to them) and spacetime coordinates (and derivatives with respect to them) commute,

$$
\begin{aligned}
\left[\hat{Q}_a, P_\mu\right] &= i\frac{\partial}{\partial\theta^a}P_\mu - iP_\mu\frac{\partial}{\partial\theta^a} = i\frac{\partial}{\partial\theta^a}i\frac{\partial}{\partial x^\mu} - ii\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a} = -\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a} \\
&= -\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial\theta^a}\frac{\partial}{\partial x^\mu} = 0.
\end{aligned}
\tag{8-76}
$$

From (8-51) and (8-19) once again,

$$
\begin{aligned}
\left[\hat{Q}_a^\dagger, P_\mu\right] &= \left(-i\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\right)P_\mu - P_\mu\left(-i\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\right) \\
&= \left(-i\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\right)i\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial x^\mu}\left(-i\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\right) \\
&= \left(\frac{\partial}{\partial\theta^{a*}}\frac{\partial}{\partial x^\mu} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right) - \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^{a*}} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right) \\
&= \left(\frac{\partial}{\partial\theta^{a*}}\frac{\partial}{\partial x^\mu} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right) - \left(\frac{\partial}{\partial\theta^{a*}}\frac{\partial}{\partial x^\mu} + \theta^b\left(\sigma^\nu\right)_{ba}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\right) = 0.
\end{aligned}
\tag{8-77}
$$

From (8-48) and (8-51), along with (8-19) again, where we note that part way through, we employ the fact that spinor derivatives anticommute,

$$
\begin{aligned}
\left[\hat{Q}_a, \hat{Q}_b^\dagger\right]_+ &= i\frac{\partial}{\partial\theta^a}\left(-i\frac{\partial}{\partial\theta^{b*}} + \theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\right) + \left(-i\frac{\partial}{\partial\theta^{b*}} + \theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\right)i\frac{\partial}{\partial\theta^a} \\
&= \frac{\partial}{\partial\theta^a}\left(\frac{\partial}{\partial\theta^{b*}} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\right) + \left(\frac{\partial}{\partial\theta^{b*}} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\right)\frac{\partial}{\partial\theta^a} \\
&= \frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^{b*}} + i\frac{\partial}{\partial\theta^a}\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial\theta^{b*}}\frac{\partial}{\partial\theta^a} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a} \\
&= \underbrace{\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^{b*}}}_{\text{cancels}} + i\frac{\partial}{\partial\theta^a}\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu} - \underbrace{\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial\theta^{b*}}}_{\text{cancels}} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a} \\
&= i\frac{\partial}{\partial\theta^a}\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a} \\
&= i\left(\frac{\partial}{\partial\theta^a}\theta^c\right)\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu} - i\theta^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial x^\mu} + i\,\theta^c\left(\sigma^\mu\right)_{cb}\underbrace{\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta^a}}_{\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial x^\mu}} \\
&= i\delta_a^c\left(\sigma^\mu\right)_{cb}\frac{\partial}{\partial x^\mu} = \left(\sigma^\mu\right)_{ab}i\frac{\partial}{\partial x^\mu} = \left(\sigma^\mu\right)_{ab}P_\mu.
\end{aligned}
\tag{8-78}
$$

9 Gauge (Vector) Supermultiplets

This section is a summary of Aitchison, Chap. 7, Sects. 7.1 and 7.2.

9.1 Introduction

All the SUSY we have done to here was for chiral supermultiplets (doublets for us, in N=1 SUSY), i.e., in SUSY space, the multiplet $(\chi, \theta)^T$. We have yet to discuss the force carrying fields, the gauge bosons. In this section we investigate them and their SUSY spartners, the gauginos.

In the bottom row, last two columns, we follow Aitchison's notation where he uses λ^α for both winos and gluinos and expects the reader to glean from context which is meant.

Wholeness Chart 9-1. Chiral and Gauge Supermultiplets (N = 1 SUSY)

Chiral SUSY Doublets	**Gauge SUSY Doublets**		
	U(1) QED	**SU(2) Weak Isospin**	**SU(3) QCD**
$\begin{bmatrix}\phi\\ \chi\end{bmatrix}=\begin{bmatrix}\text{scalar}\\ \text{LC spinor}\end{bmatrix}$ $\begin{bmatrix}\psi\\ \phi_\psi\end{bmatrix}=\begin{bmatrix}\text{RC spinor}\\ \text{scalar}\end{bmatrix}$	$\begin{bmatrix}A_\mu\\ \lambda\end{bmatrix}=\begin{bmatrix}\text{photon}\\ \text{photino}\end{bmatrix}$	$\begin{bmatrix}W^\alpha_\mu\\ \lambda^\alpha\end{bmatrix}=\begin{bmatrix}\text{W boson}\\ \text{wino}\end{bmatrix}$	$\begin{bmatrix}A^\alpha_\mu\\ \lambda^\alpha\end{bmatrix}=\begin{bmatrix}\text{gluon}\\ \text{gluino}\end{bmatrix}$

Note, one could argue that the F auxiliary field should be included (see (6-26), (6-27), and (6-28)) in the chiral multiplets, making them triplets instead of doublets. But, as we showed in (6-25), its equation of motion is $F = 0$, and it is an unobserved "phantom" field, so we ignore it. A similar situation arises for the gauge multiplet, where we again ignore a similar phantom field (the D field in (9-1) below) and talk of a gauge doublet, rather than a gauge triplet.

9.2 U(1) Gauge Field(s)

9.2.1 The Free U(1) SUSY Lagrangian

For the photon A^μ, the photino λ, and an auxiliary field D, which is a real scalar, the *free* Lagrangian is

$$\mathcal{L}^0_{\gamma\lambda} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \tfrac{1}{2}D^2 \quad (\text{free}) \quad \text{Aitchison (7.1) [106] + (7.14) [108]} \quad (9\text{-}1)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (9\text{-}2)$$

The auxiliary field D is needed for a consistent theory that includes off-shell particles (propagators). For on-shell particles it is not needed. The photino λ is a spin ½ fermion with zero charge and is massless. It has no e/m coupling, since it is not charged.

9.2.2 Gauge and Lorentz Invariance

(9-1) is symmetric under the good old U(1) global transformation of QED and under Lorentz transformation, though we won't bother to show it here.

9.2.3 SUSY Symmetry Transformations Invariance (for Free Field U(1) Theory)

As shown in Aitchison, the action found from the SUSY gauge/gaugino Lagrangian (9-1) is symmetric under the SUSY transformation set

$$\delta_\xi A^\mu = \xi^\dagger\bar{\sigma}^\mu\lambda + \lambda^\dagger\bar{\sigma}^\mu\xi \qquad \text{Aitchison (7.2) [107]} \qquad (9\text{-}3)$$

$$\delta_\xi\lambda = \tfrac{1}{2}i\sigma^\mu\bar{\sigma}^\nu\xi F_{\mu\nu} + \xi D \qquad \text{Aitchison (7.12) [107] +(7.16) [109]} \qquad (9\text{-}4)$$

$$\delta_\xi\lambda^\dagger = -\tfrac{1}{2}i\xi^\dagger\bar{\sigma}^\nu\sigma^\mu\xi F_{\mu\nu} + \xi^\dagger D \qquad \text{Aitchison (7.13) [107] +(7.16) [109]} \qquad (9\text{-}5)$$

$$\delta_\xi D = -i\left(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - \left(\partial_\mu\lambda\right)^\dagger\bar{\sigma}^\mu\xi\right) \qquad \text{Aitchison (7.15) [109].} \qquad (9\text{-}6)$$

9.2.4 The Free Plus Interacting Lagrangian (for U(1) theory)

Interactions in U(1), i.e., Abelian, SUSY theory are not discussed at this point in Aitchison. Instead, one moves on to non-Abelian theory including interactions.

9.3 SU(2) Weak Isospin Gauge Fields

We consider the three high-energy (before symmetry breaking) W_α^μ (α = 1,2,3) real SU(2) weak isospin fields and their SUSY spartners, the three spinor fields λ^α, all of which are massless (before symmetry breaking). The W_α^μ are real fields, so the λ^α are, as well. $\lambda^\alpha = \lambda^{\alpha\dagger}$.

9.3.1 The Free SU(2) Weak Isospin Lagrangian

Compare the W fields of (9-7) and (9-8) with those of Klauber, Vol. 2, pg. 176, (6-48), where $-G^i{}_{\mu\nu}$ there = $F^\alpha{}_{\mu\nu}$ here, as we are employing Aitchison's notation rather than Klauber's.

$$\mathcal{L}^0_{W\lambda} = -\frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\partial_\mu\lambda^\alpha + \frac{1}{2}D^\alpha D^\alpha \quad \text{(free)} \qquad \text{Aitchison (7.25) [110]} + D \text{ field.} \qquad (9\text{-}7)$$

$$F^\alpha_{\mu\nu} = \partial_\mu W^\alpha_\nu - \partial_\nu W^\alpha_\mu - g\varepsilon^{\alpha\beta\gamma}W^\beta_\mu W^\gamma_\mu \qquad \text{Aitchison (7.19) [109]} \qquad (9\text{-}8)$$

9.3.2 The Free Plus Interacting SU(2) Lagrangian

We substitute the covariant derivative D_μ (shown in (9-9) below) for ∂_μ in (9-7). Note that D_μ is completely different from the auxiliary field D^α. Here, it turns out (with references in Aitchison to other sources) to be (9-9), but for our purposes, we can just consider that a good guess (as it leads to a consistent theory). Note that (9-9) differs from Klauber, Vol.2, pg. 175, (6-47) in using $\varepsilon^{\alpha\beta\gamma}$ instead of σ_j, and in having both sfermion and boson factors, rather than simply a boson factor. This is simply because the cited reference was for SU(2) isospin space and here we are in SUSY space, a different kind of space.

$$\partial_\mu\lambda^\alpha \to \left(D_\mu\lambda\right)^\alpha = \partial_\mu\lambda^\alpha - g\varepsilon^{\alpha\beta\gamma}W^\beta_\mu\lambda^\gamma \qquad \text{Aitchison (7.28) [111]} \qquad (9\text{-}9)$$

With (9-9) in (9-7), we have the full gauge SUSY Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha + \frac{1}{2}D^\alpha D^\alpha \quad (\text{full } \mathcal{L}) \qquad \text{Aitchison (7.29) [111].} \quad (9\text{-}10)$$

9.3.3 Gauge and Lorentz Invariance (for Free plus Interacting SU(2) Theory)

(9-10) gives us a full Lagrangian (including all fields) that is symmetric under the electroweak local SU(2) (isospin) transformation of high-energy electroweak theory and also, under Lorentz transformation, but we won't bother to show that here.

9.3.4 SUSY Symmetry Transformation Invariance (for Free plus Interacting Field SU(2) Theory)

It turns out (we and Aitchison don't show it, though) that the action obtained from the Lagrangian (9-10) is symmetric under the following SUSY transformation set. Note the similarity to (9-3), (9-4), and (9-6).

$$\delta_\xi W^{\mu\alpha} = \xi^\dagger\bar{\sigma}^\mu\lambda^\alpha + \lambda^{\alpha\dagger}\bar{\sigma}^\mu\xi \qquad \text{Aitchison (7.30) [111]} \qquad (9\text{-}11)$$

$$\delta_\xi\lambda^\alpha = \frac{1}{2}i\sigma^\mu\bar{\sigma}^\nu\xi F^\alpha_{\mu\nu} + \xi D^\alpha \qquad \text{“} \qquad (9\text{-}12)$$

$$\delta_\xi D^\alpha = -i\left(\xi^\dagger\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha - \left(D_\mu\lambda\right)^{\alpha\dagger}\bar{\sigma}^\mu\xi\right) \qquad \text{“} \qquad . \quad (9\text{-}13)$$

9.4 SU(3) QCD Fields

Similar logic to that of Sect. 9.3 holds for other $SU(n)$ theories where $n > 2$, such as that of strong interactions.

9.5 Raising and Lowering Operators for Vector Multiplets

Aitchison does not show it, but given that (9-10) is symmetric under (9-11) to (9-13), we would expect to continue on using Noether's theorem, via the steps 3 to 7 of Sect. 6.1, pg. 25, Steps to Deduce SUSY, to find the operators that transform a vector into its sfermion partner and a spinor fermion into its gaugino partner. Surprisingly, this is not shown in Aitchison, or other SUSY texts (that I know of). We summarize those steps below.

<u>Step 3: The Commutation Relations of the Generators</u>

As with the chiral supermultiplet, finding the commutation (and anticommutation) relations for the vector supermultiplet generators is complicated and actually unnecessary for SUSY, so it is generally skipped, and we do so here. (It is a key part of SU(n) theory development, however.)

<u>Step 4: Use Noether's Theorem to Find the Conserved 4-curren(s)</u>

From (6-40), repeated below, which is the Noether current for a variation in the fields in $\mathcal{L}$ via ξ,

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\phi^r_{,\mu}}\frac{\partial\phi'}{\partial\xi} \qquad \xi \text{ real} \qquad \text{repeat of (6-40)} \qquad (9\text{-}14)$$

where ξ is a two component Grassmann variable, more precisely expressed as ξ_a, where a = 1,2. Following steps similar to those after (6-40), we can then find, where the subscript "vec" means we are working with the Lagrangian for the vector (gauge) supermultiple fields,

$$ {}_{vec}j^{\mu}_{SUSY} = {}_{vec}J^{\mu}_{a} \qquad a = 1,2\,. \tag{9-15}$$

We don't actually do this here. Our goal is to understand the (lengthy) steps involved, so we can feel some level of comfort with the final results, which we will just state.

Step 5: Find the Conserved Charge Operators

We then find the conserved charges ("charge operators", to be precise)

$$ {}_{vec}Q_1 = \int J^0_1 d^3x \qquad\qquad {}_{vec}Q_2 = \int J^0_2 d^3x\,. \tag{9-16}$$

Again, we don't do this explicitly here, but we would find the ${}_{vec}Q_a$ contain certain construction and destruction operator fields for vectors and spinors.

Note that the particular construction and destruction operators in each of ${}_{vec}Q_1$ and ${}_{vec}Q_2$ would determine the action of those charge operators on states. It turns out that ${}_{vec}Q_1$, like Q_1 for chiral multiplet (doublet), lowers spin by ½. It converts a spin 1 (vector) state into a spin ½ fermion state. ${}_{vec}Q_2$, like Q_2, raises spin by ½ and converts a spin −1 state into a spin −½ state.

In parallel with Q_a, both ${}_{vec}Q_a$ take a higher spin magnitude state to a lower spin magnitude state, and their complex conjugates do the reverse.

Step 6: Determine the Commutation Relations for the ${}_{vec}Q_a$

Using (6-57) and, instead of (6-58) for scalar fields, the following for vector fields,

$$\left[W^{\alpha}_{\mu}(t,\mathbf{x}), W^{\beta\dagger}_{\nu}(t,\mathbf{y})\right] = ig_{\mu\nu}\delta^{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})\,, \tag{9-17}$$

one can find the commutation, and anticommutation, relations for the charges ${}_{vec}Q_a$. These parallel what we've seen before for the chiral supermultiplet (6-9),

$$\begin{aligned}
&\left[{}_{vec}Q_a, {}_{vec}Q^{\dagger}_b\right]_+ = (\sigma^{\mu})_{ab}P_{\mu} \\
&\left[{}_{vec}Q_a, {}_{vec}Q_b\right]_+ = 0 \\
&\left[{}_{vec}Q_a, P_{\mu}\right] = \left[{}_{vec}Q^{\dagger}_a, P_{\mu}\right] = 0.
\end{aligned} \tag{9-18}$$

Step 7: Determine what Effect Each ${}_{vec}Q_a$ Has on States

We already found the effect of each ${}_{vec}Q_a$ on states, to some degree, in Step 5, but, similar to what we did with the chiral supermultiplet charges, we can learn more from (9-18).

In particular, in parallel with (6-75) and (6-76), we can use those relations to show that, under either of ${}_{vec}Q_a$, the 4-momentum and the mass (which could be zero) of the original spin 1 state remains unchanged.

Summary

Thus, in parallel with Wholeness Chart 6-1, pg. 25, we can summarize as follows. Note that, in the chart, the particles with opposite spins also have opposite 3-momenta, so their helicity is the same.

Wholeness Chart 9–2. The Effects of the Two Vector Multiplet SUSY Charges on States

Spin	Particle	${}_{vec}Q_1$	${}_{vec}Q_1^{\dagger}$	${}_{vec}Q_2$	${}_{vec}Q_2^{\dagger}$	Antiparticle	${}_{vec}Q_1$	${}_{vec}Q_1^{\dagger}$	${}_{vec}Q_2$	${}_{vec}Q_2^{\dagger}$
+ 1	LH boson					RH anti-boson				
		↓	↑						↑	↓
+½	LH fermion					RH anti- fermion				
−½	LH fermion					RH anti- fermion				
				↑	↓		↓	↑		
−1	LH boson					RH anti-boson				

(Similar effects for RH vector bosons and LH anti-vector bosons)

NOTE: Many authors treat Q_a and ${}_{vec}Q_a$ as essentially the same thing, i.e., use Q_a as an operator on both the SUSY chiral doublet and the vector (gauge) doublet.

10 Combining Chiral and Gauge Supermultiplets

This Section overviews Chap. 3, Sect. 7.3 of Aitchison.

We have looked at chiral multiplets and gauge multiplets separately, including interactions within each type multiplet between fields in that multiplet. Now, we have to examine interactions between fields in one multiplet with fields in the other. This can be done in three steps, one for each of the gauge multiplet field types.

1. Gauge fields (vectors A_μ, $W^\alpha{}_\mu$, and $A^\alpha{}_\mu$,) coupling to chiral multiplet fields (scalars ϕ_i, fermions χ_i, and auxiliary fields F^α).
2. Gaugino fields (λ [photino], λ^α [wino], and λ^α [gluino]) coupling to chiral multiplet fields (ϕ_i, χ_i, and F^α).
3. The gauge auxiliary fields D^α coupling to chiral multiplet fields (ϕ_i, χ_i, and F^α).

For simplicity, this will be done first in Sect. 10.1 for Abelian ($U(1)$) theory with a single vector supermultiplet and a single chiral multiplet, such as a photon multiplet and an electron multiplet. Then, this will be extended in Sect. 10.2 to i) non-Abelian ($SU(n)$) theory and ii) several vector supermultiplets and several chiral supermultiplets.

10.1 $U(1)$ Interactions between Vector and Chiral Supermultiplets

$U(1)$ interactions in the standard model (SM), whether QED or hypercharge, have only one vector field (A_μ in QED or B_μ or hypercharge above Higgs symmetry breaking energies). So, in the following, we don't have to worry about more than one such field. Each fermion in such theories would be coupled to the single vector/gauge boson, albeit with possibly different coupling constant parameters. Hence, we can model $U(1)$ theories with a single gauge field, represented by A_μ and the field χ, where χ could be an electron, a muon, a neutrino, etc.

From (6-24) for the free chiral supermultiplet fields and (9-1) for the free gauge supermultiplet fields, we have the free Lagrangian including both the chiral and gauge multiplet fields,

$$\mathcal{L} = \underbrace{\partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F}_{\mathcal{L}^0_{\chi\phi}} \underbrace{- \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + \tfrac{1}{2} D^2}_{\mathcal{L}^0_{\gamma\lambda}} . \tag{10-1}$$

We then substitute the covariant derivative for the single gauge field,

$$D_\mu = \partial_\mu + iqA_\mu , \qquad \text{Aitchison (7.31) [112]} \tag{10-2}$$

into (10-1), to get

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu D_\mu \chi + F^\dagger F - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + \tfrac{1}{2} D^2 . \qquad \text{Aitchison (7.32) [112]} \tag{10-3}$$

Note there is ∂_μ instead of D_μ in the next to last term because, as we noted earlier, λ has no charge and so doesn't interact with the A_μ field of (10-2).

While (10-3) includes interactions of ϕ with A_μ and χ with A_μ, it does not include interactions between ϕ, χ, λ, F, and D, so we need to include those. Any such terms must be Lorentz-invariant, $U(1)$ gauge-invariant, SUSY invariant, and renormalizable. Any potential interaction terms that are not can be thrown out.

Aitchison, on pgs. 112 to 116 considers possible interaction terms and eliminates all that don't meet these criteria to arrive at

$$\begin{aligned}\mathcal{L} = {} & D_\mu \phi^\dagger D^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu D_\mu \chi + F^\dagger F - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + \tfrac{1}{2} D^2 \\ & - \sqrt{2}\left(\left(\phi^\dagger \chi \right) \cdot \lambda + \lambda^\dagger \cdot \left(\chi^\dagger \phi \right) \right) \quad - q\phi^\dagger \phi D.\end{aligned} \qquad \text{Aitchison (7.63) [116]} \tag{10-4}$$

Using the Euler-Lagrange equation for the D field with (10-4), we have

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial D_{,\mu}} \right) - \frac{\partial \mathcal{L}}{\partial D} = 0 \quad \rightarrow \quad 0 - D + q\phi^\dagger \phi = 0 \quad \rightarrow \quad D = q\phi^\dagger \phi , \qquad \text{Aitchison (7.64) [117]} \tag{10-5}$$

which, when used in (10-4), leaves us with

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu D_\mu \chi + F^\dagger F - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda$$
$$- \sqrt{2}\left(\left(\phi^\dagger \chi \right) \cdot \lambda + \lambda^\dagger \cdot \left(\chi^\dagger \phi \right) \right) \quad - \tfrac{1}{2} q^2 \left(\phi^\dagger \phi \right)^2 . \tag{10-6}$$

The first two terms in the second row of (10-6) represent the interaction between ϕ, χ, and λ. Note it is one term plus its Hermitian conjugate.

The last term in (10-6) is of the form we found in electroweak theory for the Higgs field before symmetry breaking. But, whereas the constant (symbol λ there, which is quite different from the photino λ symbol here) for that term in that theory was indeterminate (via the theory itself), here, the theory provides a value of $\frac{1}{2}q^2$, where q is the gauge coupling constant.

10.2 *SU*(*n*) Interactions between Several Vector and Chiral Supermultiplets

In considering non-Abelina interactions between several different scalar, spinor, and vector fields, we proceed as we did in Sect. 10.1 for Abelian fields and one scalar, one spinor, and one vector. That is, we start with the free field Lagrangian, then substitute the covariant derivative (to get interaction terms between the chiral multiplet fields and the vector gauge bosons). Then, we determine the remaining interaction terms between the ϕ_i, χ_i, λ^α, F_i, and D^α fields, by restricting all potential terms to be Lorentz invariant, gauge invariant, SUSY invariant, and renormalizable.

We start with the chiral multiplet Lagrangian in the W-Z model, where the free Lagrangian, including the superpotential, is (7-1) plus (7-7) with (7-3) and (7-4), all repeated below.

$$\mathcal{L}_{\text{free WZ}} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i \bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i \qquad \mathcal{L}_{\text{int WZ}} = W_i F_i - \tfrac{1}{2} W_{ij} \chi_j \cdot \chi_j + h.c. \qquad \text{repeat of (7-1) \& (7-7)} \tag{10-7}$$

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad W_{ij} = \frac{\partial^2 W}{\partial \phi \partial \phi_j} \qquad \text{repeat of (7-3) \& (7-4)} \tag{10-8}$$

$$\begin{aligned} \mathcal{L}_{WZ} &= \mathcal{L}_{\text{free WZ}} + \mathcal{L}_{\text{int WZ}} \\ &= \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i \bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i + \left\{ \frac{\partial W}{\partial \phi_i} F_i - \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \, \partial \phi_j} \chi_i \cdot \chi_j + h.c. \right\} . \end{aligned} \qquad \text{Aitchison (7.66) [117]} \tag{10-9}$$

We then employ the covariant derivative for the ϕ_i and χ_i fields, where T^α are the generators of the gauge group. For *SU*(2) theory $T^\alpha = \frac{1}{2}\sigma^\alpha$, where σ^α are the Pauli matrices; and for *SU*(3) theory, they are the Gell-Mann matrices divided by 2.

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu^\alpha T^\alpha \quad \text{for both } \phi_i \text{ and } \chi_i \qquad \text{see Aitchison (7.67) and (7.68) [117].} \tag{10-10}$$

For *SU*(2) theory, g would be the weak coupling constant; for *SU*(3) theory, the QCD coupling constant.

With (10-10) in (10-9), we have

$$\mathcal{L}_{WZ\,covariantized} = D_\mu \phi_i^\dagger D^\mu \phi_i + \chi_i^\dagger i \bar{\sigma}^\mu D_\mu \chi_i + F_i^\dagger F_i + \left\{ \frac{\partial W}{\partial \phi_i} F_i - \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \chi_i \cdot \chi_j + h.c. \right\} . \tag{10-11}$$

This is the chiral covariantized (employs covariant derivative) Lagrangian. To it, in order to get the complete Lagrangian, we will need to add i) the gauge covariantized Lagrangian (9-10), repeated below, plus ii) Lorentz and gauge invariant renormalizable interactions between the ϕ_i, χ_i, λ^α, F_i, and D^α fields.

$$\mathcal{L}_{gauge\ covariantized} = -\tfrac{1}{4} F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} + i\lambda^{\alpha\dagger} \bar{\sigma}^\mu \left(D_\mu \lambda \right)^\alpha + \tfrac{1}{2} D^\alpha D^\alpha \qquad \text{repeat of (9-10)} \tag{10-12}$$

Aitchison (pgs. 118-119) examines all possible Lorentz and gauge invariant renormalizable interactions between the ϕ_i, χ_i, λ^α, F_i, and D^α fields, which could be added to (10-11) and (10-12) to get the complete Lagrangian. Upon doing this, he finds they equal the second row in (10-13) below, and thus arrives at a complete Lagrangian of

$$\begin{aligned} \mathcal{L} = {} & \mathcal{L}_{WZ\,covariantized} \\ & + \mathcal{L}_{gauge\ covariantized} \\ & - \sqrt{2} g \left(\left(\phi_i^\dagger T^\alpha \chi_i \right) \cdot \lambda^\alpha + \lambda^{\alpha\dagger} \cdot \left(\chi_i^\dagger T^\alpha \phi_i \right) \right) - g \left(\phi_i^\dagger T^\alpha \phi_i \right) D^\alpha \end{aligned} \qquad \text{Aitchison (7.72) [119],} \tag{10-13}$$

more transparently expressed as

$$\mathcal{L} = D_\mu\phi_i^\dagger D^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu D_\mu\chi_i + F_i^\dagger F_i + \left\{\frac{\partial W}{\partial\phi_i}F_i - \tfrac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i\cdot\chi_j + h.c.\right\}$$
$$-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha + \tfrac{1}{2}D^\alpha D^\alpha \tag{10-14}$$
$$-\sqrt{2}g\left(\left(\phi_i^\dagger T^\alpha\chi_i\right)\cdot\lambda^\alpha + \lambda^{\alpha\dagger}\cdot\left(\chi_i^\dagger T^\alpha\phi_i\right)\right) - g\left(\phi_i^\dagger T^\alpha\phi_i\right)D^\alpha .$$

Using the Euler-Lagrange equation in D^α, parallel to what we did in (10-5), we find

$$D^\alpha = g\sum_j \phi_j^\dagger T^\alpha\phi_j , \qquad \text{Aitchison (7.73) [119]}, \tag{10-15}$$

which, when substituted in (10-14), leaves us with

$$\mathcal{L} = D_\mu\phi_i^\dagger D^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu D_\mu\chi_i + F_i^\dagger F_i + \frac{\partial W}{\partial\phi_i}F_i + F_i^\dagger\left(\frac{\partial W}{\partial\phi_i}\right)^\dagger - \left\{\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i\cdot\chi_j + h.c.\right\}$$
$$-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha - \sqrt{2}g\left(\left(\phi_i^\dagger T^\alpha\chi_i\right)\cdot\lambda^\alpha + \lambda^{\alpha\dagger}\cdot\left(\chi_i^\dagger T^\alpha\phi_i\right)\right) - \tfrac{1}{2}g^2\left(\phi_i^\dagger T^\alpha\phi_i\right)\left(\phi_j^\dagger T^\alpha\phi_j\right). \tag{10-16}$$

With the Euler-Lagrange equation again for F_i, we find

$$\frac{\partial W}{\partial\phi_i} = -F_i^\dagger \quad\to\quad \left(\frac{\partial W}{\partial\phi_i}\right)^\dagger = -F_i . \tag{10-17}$$

Substituting (10-17) in (10-16) and re-arranging (10-16), we get

$$\mathcal{L} = D_\mu\phi_i^\dagger D^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu D_\mu\chi_i + \overbrace{F_i^\dagger F_i - F_i^\dagger F_i}^{0} + \left\{-\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i\cdot\chi_j + h.c.\right\}$$
$$-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha - \sqrt{2}g\left(\left(\phi_i^\dagger T^\alpha\chi_i\right)\cdot\lambda^\alpha + \lambda^{\alpha\dagger}\cdot\left(\chi_i^\dagger T^\alpha\phi_i\right)\right) \tag{10-18}$$
$$\underbrace{-F_i^\dagger F_i - \tfrac{1}{2}g^2\left(\phi_i^\dagger T^\alpha\phi_i\right)\left(\phi_j^\dagger T^\alpha\phi_j\right)}_{-\mathcal{V}\left(\phi_i,\phi_i^\dagger\right)},$$

where we have isolated what is effectively the potential $\mathcal{V}$ of the scalar field(s).

It is implicit in (10-18) that there is only one gauge field, presumably either the three $W_\mu{}^\alpha$ of $SU(2)$ theory, the eight gluons of $SU(3)$ theory, or the B_μ of $U(1)$ electroweak high-energy theory. So, we can generalize (10-18), where the subscript G designates the particular gauge fields, repeated G implies summation over gauge field types, and we use (10-17) in the last line to get

$$\mathcal{L} = D_\mu\phi_i^\dagger D^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu D_\mu\chi_i + \left\{-\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i\cdot\chi_j + h.c.\right\}$$
$$-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha - \sqrt{2}g_G\left(\left(\phi_i^\dagger T_G^\alpha\chi_i\right)\cdot\lambda^\alpha + \lambda^{\alpha\dagger}\cdot\left(\chi_i^\dagger T_G^\alpha\phi_i\right)\right) \tag{10-19}$$
$$-\left|\frac{\partial W}{\partial\phi_i}\right|^2 - \tfrac{1}{2}g_G^2\left(\phi_i^\dagger T_G^\alpha\phi_i\right)\left(\phi_j^\dagger T_G^\alpha\phi_j\right).$$

With (7-2),

$$W = \tfrac{1}{2}M_{ij}\phi_i\phi_j + \tfrac{1}{6}y_{ijk}\phi_i\phi_j\phi_k \qquad \text{repeat of (7-2)}, \tag{10-20}$$

in (10-19), we find

$$\mathcal{L} = D_\mu\phi_i^\dagger D^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}^\mu D_\mu\chi_i + \left\{-\tfrac{1}{2}\tilde{M}_{ij}\chi_i\cdot\chi_j - \tfrac{1}{2}\tilde{Y}_{ijk}\chi_i\cdot\chi_j\phi_k + h.c.\right\}$$
$$-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + i\lambda^{\alpha\dagger}\bar{\sigma}^\mu\left(D_\mu\lambda\right)^\alpha - \sqrt{2}g_G\left(\left(\phi_i^\dagger T_G^\alpha\chi_i\right)\cdot\lambda^\alpha + \lambda^{\alpha\dagger}\cdot\left(\chi_i^\dagger T_G^\alpha\phi_i\right)\right) \tag{10-21}$$
$$\underbrace{-\hat{M}_{ij}^2\phi_i^\dagger\phi_j - \left(\hat{Y}_{ij}\phi_i\phi_j\right)^\dagger\left(\hat{Y}_{kl}\phi_k\phi_l\right) - \tfrac{1}{2}g_G^2\left(\phi_i^\dagger T_G^\alpha\phi_i\right)^\dagger\left(\phi_j^\dagger T_G^\alpha\phi_j\right)}_{-\mathcal{V}\left(\phi_i,\phi_i^\dagger\right)}$$

Note again, that somewhat similar to the $U(1)$ case of Sect. 10.1, the coefficient of the last ϕ^4 term is fixed by the theory, though the coefficient of the term before that in ϕ^4 has coefficient(s) that are not fixed by theory.

<u>Bottom line:</u> (10-21) is the complete SUSY Lagrangian (for Wess-Zumino theory) including the chiral supermultiplet, the gauge supermultiplet, and all interactions between all of the fields therein, for all particle types in the SUSY extension of the SM.

The gauge field ($U(1)$, $SU(2)$, and $SU(3)$) interactions with other fields are in the covariant derivative D_μ of (10-21). Interactions between other fields are found in the other terms, such as the last term in the second row, which, represented in Feynman diagrams, would have a photino λ^α, a LC fermion χ_i, and a scalar ϕ_i at a vertex.